A New Kind of Weighted (0;0,2) Interpolation on Laguerre Polynomial

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Abstract: ***In this paper we consider a special problem of mixed type (0;0,2) interpolationon on the two sets of the nodes of Laguerre polynomial in which one set consists of the nodes of $L_n^k(x)$ while the other are nodes of $L_n^{k-1}(x)$. We prove the existence, uniqueness and explicit representation of fundamental polynomials on infinite interval.***

Keywords: lacunary interpolation, Pál - Type interpolation, Laguerre Polynomial, Birkoff quadrature formula MSC 2000: 41 A 05 65 D 32

1. Introduction

In 1975 Pál [10] proved that for a given arbitrary numbers $\{\alpha_i\}_{i=1}^{n}$ and $\{\beta_i\}_{i=1}^{n-1}$ there exists a unique polynomial of degree ≤ 2n-1 satisfying the condition:

(1.1) $R_n(\xi_i) = \alpha_i \,, i = 1(1)n, \quad R_n(\xi_i) = \beta_i \,, i = 1(1)n-1$
and an initial condition $R_n(a) = 0$, where a is a given point different from the nodal points . Balázs. J.[2][3] and Szili [13] have studied problems for weighted (0,2) interpolation .T.F. XIE [15],Mathur P. and Datta S. [8] and many others mathematicians [1][4][6][7][9][11][12][14] have discussed about interpolation problems when the values of the function and its consecutive derivatives are prescribed at the given set of the points. Lénárd M. [5] investigated the Pál – type interpolation problem on the nodes of Laguerre abscissas. In Pál – type interpolation the derivative values are prescribed at the zeroes of $\omega'(x)$ while the function values are prescribed at the zeroes of $\omega_n(x) = (x - x_1) \dots (x - x_n)$. The object of this paper is to study a special problem of mixed type weighted (0;0,2) interpolation on the nodes of Laguerre polynomial .In this paper we consider the problem if $\{\xi_i\}_{i=1}^{n}$ and $\{\xi_i^*\}_{i=1}^{n}$ be the two sets of interscaled nodal points

(1.2) $0 \le \xi_0 < \xi_1^* < \xi_1 < \cdots < \xi_{n-1} < \xi_n^* < \xi_n < \infty$
on the interval [0,∞) . We seek to determine a polynomial $R_n(x)$ of minimal possible degree 3n+k satisfying the interpolatory conditions:

(1.3) $R_n(\xi_i) = g_i, \qquad R_n(\xi_i^*) = g_i^*, \qquad (\omega R_n)''(\xi_i^*) = g_i^{**}, for\ i = 1(1)n$

(1.4) $R_n^{(j)}(\xi_0) = g_0^{(j)}, j = 0,1, \dots, k$

where g_i, g_i^*, $g_i^{**}$and $g_0^{(j)}$ are arbitrary real numbers. In general the problem is not regular when Laguerre polynomials $L_n^{(k)}(x)$ and $L_n^{(k-1)}(x)$ have zeroes $\{\xi_i\}_{i=1}^{n}$ and $\{\xi_i^*\}_{i=1}^{n}$ respectively and $x_0 = 0$ but it becomes regular with weight function $\omega(x) = e^{-x/2}x^{k/2}$. We prove existence, uniqueness and explicit representation of fundamental polynomials.

2. Preliminaries

In this section we shall give some well-known results which are as follws:
As we know that the Laguerre polynomial is a constant multiple of a confluent hypergeometric function so the differential equation is given by

(2.1) $xD^2L_n^k(x) + (1 + k - x)DL_n^k(x) + nL_n^k(x) = 0$

(2.2) $L_n^{(k-1)'}(x) = -L_{n-1}^{(k)}(x)$

Also using the identities
(2.3) $L_n^{(k)}(x) = L_n^{(k+1)}(x) - L_{n-1}^{(k+1)}(x)$

(2.4) $xL_n^{(k)'}(x) = nL_n^{(k)}(x) - (n + k)L_{n-1}^{(k)}(x)$

We can easily find a relation

(2.5) $\frac{d}{dx}[x^k L_n^k(x)] = (n + k)x^{k-1}L_n^{(k-1)}(x)$

By the following conditions of orthogonality and normalization we define Laguerre polynomial $L_n^{(k)}(x), for\ k > -1$

(2.6)
$\int_0^\infty e^{-x}x^k L_n^{(k)}(x)L_m^{(k)}(x)dx = \Gamma k + 1 \binom{n+k}{n}\delta_{nm}\ n, m =$ *0,1,2,.....*

(2.7) $L_n^{(k)}(x) = \sum_{\mu=0}^{n}\binom{n+k}{n}\frac{(-x)^\mu}{\mu!}$

The fundamental polynomials of Lagrange interpolation are given by

(2.8) $l_j(x) = \frac{L_n^{(k)}(x)}{L_n^{(k)'}(x_j)(x-x_j)} = \delta_{i,j}$

(2.9) $l_j^*(x) = \frac{L_n^{(k-1)}(x)}{L_n^{(k-1)'}(y_j)(x-y_j)} = \delta_{i,j}$

(2.10) $l_j^{*'}(y_j) = \begin{cases} \frac{L_n^{(k-1)'}(y_i)}{L_n^{(k-1)'}(y_j)(y_i-y_j)} & i \neq j \\ -\frac{(k-y_j)}{2y_j} & i = j \end{cases} \quad i,j = 1(1)n$

(2.11)

$l_j^{*''}(y_j) = \begin{cases} -\frac{L_n^{(k-1)'}(y_i)}{L_n^{(k-1)'}(y_j)(y_i-y_j)}\left[\frac{(k-y_i)}{y_i} + \frac{2}{(y_i-y_j)}\right] & i \neq j \\ \frac{(k-y_j)^2-(ny_j-k)}{3y_j^2} & i = j \end{cases} \quad i,j = 1(1)n$

(2.12) $l_j'(y_j) = \frac{1}{(y_j-x_j)}\left[\frac{L_n^{(k)'}(y_j)}{L_n^{(k)'}(x_j)} - \frac{L_n^{(k)}(y_j)}{L_n^{(k)'}(x_j)(y_j-x_j)}\right], j = 1(1)n$

3. New Result

Theorem 1: For n >1 fixed integer let $\{g_i\}_{i=1}^n$, $\{g_i^*\}_{i=1}^n$, $\{g_i^{**}\}_{i=1}^n$ and, $\{g_0^{(j)}\}_{j=0}^k$ are arbitrary real numbers then there exists a unique polynomial $R_n(x)$ of minimal possible degree $\leq$ 3n+k on the nodal points (1.2) satisfying the condition (1.3) and (1.4).

The polynomial $R_n(x)$ can be written in the form (3.1) $\boldsymbol{R_n(x) = \sum_{j=1}^n U_j(x)g_j + \sum_{j=1}^n V_j(x)g_j^* + \sum_{j=1}^n W_j(x)g_j^{**} +}$ ***j=0kCj(x)g0(j)***

Where $U_j(x)$, $V_j(x)$, $W_j(x)$ and $C_j(x)$ are fundamental polynomials of degree $\leq$ 3n+k
given by

(3.2) $U_j(x) = \frac{1}{x_j^{(k+1)}L_n^{(k-1)}(x_j)}\left[x^{k+1}l_j(x)L_n^{(k-1)}(x) - \frac{x^k L_n^{(k)}(x)L_n^{(k-1)}(x)}{L_n^k(y_j)L_n^{(k)'}(x_j)}\int_0^x \frac{x_j L_n^{(k-1)'}(t)-[x_j-(n+k)(t-x_j)]L_n^{(k-1)}(t)}{(t-x_j)^2}dt\right]$

(3.3) $V_j(x) = \frac{1}{y_j^{(k+1)}[L_n^{(k)}(y_j)]^2}\left[x^{k+1}[L_n^{(k)}(x)]^2 l_j^*(x) + \frac{\eta x^k L_n^{(k)}(x)L_n^{(k-1)}(x)}{L_n^{(k-1)'}(y_j)}\int_0^x \frac{(t-y_j)L_n^{(k)}(t)+L_n^{(k-1)}(t)}{(t-y_j)^2}dt\right]$

(3.4) $W_j(x) = -\frac{e^{y_j/2}x^k L_n^{(k)}(x)L_n^{(k-1)}(x)}{2y_j^{3k/2}[L_n^{(k)}(y_j)]^2}\int_0^x l_j^*(t)dt$

(3.5)
$C_j(x) = p_j(x)x^j L_n^{(k)}(x)\left[L_n^{(k-1)}(x)\right]^2 + x^k L_n^{(k)}(x)L_n^{(k-1)}(x)\left[c_j - \int_0^x \frac{L_n^{(k)'}(t)p_j(t)+q_j(t)L_n^{(k)}(t)}{t^{k-j}}dt\right]$
$, j = 0,1,\dots,k-1$

(3.6) $C_k(x) = \frac{1}{\binom{n+k}{k}k![L_n^{(k-1)}(0)]^2}x^k L_n^{(k)}(x)[L_n^{(k-1)}(x)]^2$

Where $U_j(x)$, $V_j(x)$, $W_j(x)$and $C_j(x)$ are fundamental polynomials of degree $\leq$ 3n+k . $p_j(x)$ and $q_j(x)$ are polynomials of degree at most k-j-1 . c_j is defined in (4.21) and η is defined as:

(3.7) $\eta = \frac{1}{(k+y_j)}\left[\frac{1}{12}(k^2+y_j{}^2) - \frac{k}{6}(13y_j+7) - \frac{7}{3}ny_j\right]$

4. Proof of Theorem 1

Let $U_j(x)$, $V_j(x)$, $W_j(x)$ and $C_j(x)$ are polynomials of degree $\leq$ 3n+k satisfying conditions (4.1), (4.2), (4.3) and (4.4) respectively.

(4.1) $\begin{cases} U_j(x_i) = \delta_{i,j} \\ U_j(y_i) = 0 \\ [e^{-x/2}x^{k/2}U_j(x)]''_{x=y_i} = 0, i = 1(1)n \text{ and } l = \\ U_j^{(l)}(0) = 0 \end{cases}$
$0,1,\dots,k$

(4.2) $\begin{cases} V_j(x_i) = 0 \\ V_j(y_i) = \delta_{i,j} \\ [e^{-x/2}x^{k/2}V_j(x)]''_{x=y_i} = 0, i = 1(1)n \text{ and } l = \\ V_j^{(l)}(0) = 0 \end{cases}$
$0,1,\dots,k$

(4.3) $\begin{cases} W_j(x_i) = 0 \\ W_j(y_i) = 0 \\ [e^{-x/2}x^{k/2}W_j(x)]''_{x=y_i} = \delta_{i,j} \\ W_j^{(l)}(0) = 0 \end{cases}$

(4.4) $\begin{cases} C_k(x_i) = 0 \\ C_k(y_i) = 0 \\ [e^{-x/2}x^{k/2}C_k(x)]''_{x=y_i} = 0, i = 1(1)n \text{ and } l = \\ C_k^{(l)}(0) = \delta_{l,k} \end{cases}$
$0,1,\dots,k$

To determine $U_j(x)$ let
(4.5)
$U_j(x) =$
$C_1 x^{k+1}l_j(x)L_n^{(k-1)}(x) +$
$C_2 x^k L_n^{(k)}(x)L_n^{(k-1)}(x)\int_0^x \frac{L_n^{(k-1)'}(t)+[C_3+C_4(t-x_j)]L_n^{(k-1)}(t)}{(t-x_j)^2}dt$

Where C_1, C_2, C_3 and C_4 are constants . $l_j(x)$ is defined in (2.8) . As $U_j(x)$ is a polynomial of degree $\leq$ 3n+k so the integrand in (4.5) must be a polynomial of at most degree n which implies

(4.6) $L_n^{(k-1)'}(t) + [C_3 + C_4(t-x_j)]L_n^{(k-1)}(t) = 0$

By using (2.2), (2.3), (2.8) we determine

(4.7) $C_3 = -1$, $C_4 = \frac{n+k}{x_j}$ $C_1 = \frac{1}{x_j^{(k+1)}L_n^{(k-1)}(x_j)}$

Since $U_j(x)$ satisfies the conditions (4.1) by which we obtain

(4.8) $C_2 = -\frac{1}{x_j^k L_n^k(y_j)L_n^{(k)'}(x_j)L_n^{(k-1)}(x_j)}$

Hence we find the first fundamental polynomial $U_j(x)$of degree $\leq$ 3n+k

To find second fundamental polynomial let

(4.9) $V_j(x) = C_5 x^{k+1}[L_n^{(k)}(x)]^2 l_j^*(x) +$
$+C_6 x^k L_n^{(k)}(x)L_n^{(k-1)}(x)\int_0^x \frac{(t-y_j)L_n^{(k)}(t) + C_7 L_n^{(k-1)}(t)}{(t-y_j)^2}dt$

Where C_5, C_6, C_7 and C_8 are arbitrary constants. Since this is a polynomial of degree $\leq$ 3n+k therefore the expression under the integral sign must be a polynomial of degree n, which implies

(4.10) $C_7 = 1$

By using (2.9), first and second condition of (4.2) we get

(4.11) $C_5 = \frac{1}{y_j^{(k+1)}[L_n^{(k)}(y_j)]^2}$

Using third condition of (4.2) we get

(4.12) $C_6 = \frac{\eta}{y_j^{(k+1)}[L_n^{(k)}(y_j)]^2 L_n^{(k-1)'}(y_j)}$

where $\eta = \frac{1}{(k+y_j)}[\frac{1}{12}(k^2+y_j^2) - \frac{k}{6}(13y_j+7) - \frac{7}{3}ny_j]$

Hence we find the first fundamental polynomial $V_j(x)$of degree ≤ 3n+k

Again let

(4.13) $W_j(x) = C_8 x^k L_n^{(k)}(x) L_n^{(k-1)}(x) \int_0^x l_j^*(t)dt$

Where C_9 is a constant,$l_j^*(t)$ is defined in (2.9) .$W_j(x)$ is polynomial of degree ≤ 3n+k satisfying the conditions (4.3) by which we obtain

(4.14) $C_8 = -\frac{e^{y_j/2}}{2y_j^{3k/2}[L_n^{(k)}(y_j)]^2}$

Hence we find the third fundamental polynomial $W_j(x)$ of degree ≤ 3n+k

To find $C_j(x)$, we assume $C_j(x)$ for fixed $j \in \{0,1,\dots\dots,k-1\}$ in the form

(4.15) $C_j(x) = p_j(x)x^j L_n^{(k)}(x)[L_n^{(k-1)}(x)]^2 + x^k L_n^{(k)}(x)L_n^{(k-1)}(x)g_n(x)$

Where $p_j(x)$ and $g_n(x)$ are polynomials of degree k-j-1 and n respectively . Now it is clear that $C_j^{(l)}(0) = 0\ for\ (l = 0,\dots\dots,j-1)$ and since $L_n^{(k)}(x_i) = 0$ and $L_n^{(k-1)}(y_i) = 0$ we get

$C_j(x_i) = 0\ and\ C_j(y_i) = 0\ for\ i = 1(1)n$. The coefficient of the polynomial $p_j(x)$ are calculated by the system

(4.16) $C_j^{(l)}(0) = \frac{d^l}{dx^l}[p_j(x)x^j L_n^{(k)}(x)[L_n^{(k-1)}(x)]^2]_{x=0} = \delta_{i,j}\ \ (l = j,\dots\dots,k-1)$

now from the equation $C_j^{(k)}(0) = 0$ we get

(4.17) $c_j = g_n(0) = \frac{-1}{\binom{n+k}{k}k!L_n^{(k-1)}(0)}\frac{d^k}{dx^k}[p_j(x)x^j L_n^{(k)}(x)[L_n^{(k-1)}(x)]^2]_{x=0}$

Now using the condition $[e^{-x/2}x^{k/2}C_j(x)]''_{x=y_i} = 0$ of (4.7), we get

(4.18) $g_n'(y_i) = -(y_i)^{j-k}L_n^{(k-1)'}(y_i)p_j(y_i)$

Which implies $g_n'(x)$ as follows

(4.19) $g_n'(x) = -\frac{L_n^{(k-1)'}(x)p_j(x) + q_j(x)L_n^{(k-1)}(x)}{x^{k-j}}$

Where $q_j(x)$ is a polynomial of degree k-j-1 and function $g_n'(x)$ will be a polynomial iff for $r = 0,1,\dots,k-j-1$

(4.20) $\frac{d^r}{dx^r}[L_n^{(k-1)'}(x)p_j(x) + q_j(x)L_n^{(k-1)}(x)]_{x=0} = 0$

The coefficients of $q_j(x)$ are uniquely calculated by this system . now integrating (4.19) we get

(4.21) $g_n(x) = g_n(0) + \int_0^x g_n'(t)dt$

Using (4.15) and (4.17) we obtain $C_j(x)$ of degree ≤ 3n+k satisfying the conditions (4.4)

Uniqueness and Existence

As $R_n(x)$ in (3.1) satisfies the conditions (1.3) and (1.4) so the existence part is proved. Now we seek to determine a polynomial $R_n(x)$ of minimal possible degree ≤ 3n+k satisfying the conditions (4.22) for $i = 1(1)n$ and $l = 0,1,\dots,k$

(4.22) $\begin{cases} S_n(x_i) = 0 \\ S_n(y_i) = 0 \\ [e^{-x/2}x^{k/2}S_n(x)]''_{x=y_i} = 0, i = 1(1)n\ and\ l = 0,1,\dots,k \\ S_n^l(0) = 0 \end{cases}$

Let us consider

(4.23) $S_n(x) = x^k L_n^{(k)}(x)L_n^{(k-1)}(x)s_n(x)$

Where $s_n(x)$ is a polynomial of at most degree n. now from the third condition of (4.22) we have

(4.24)
$[e^{-x/2}x^{k/2}S_n(x)]''_{x=y_i} = 2e^{-y_i/2}y_i^{3k/2}L_n^{(k)'}(y_i)L_n^{(k-1)'}(y_i)s_n'(y_i) = 0$

This implies $s_n'(y_i) = 0\ for\ i = 1(1)n$ so that $s_n'(x) \equiv 0$ hence $s_n(x) \equiv c$. so

(4.25) $S_n(x) = c\, x^k L_n^{(k)}(x)L_n^{(k-1)}(x)$, but

(4.26) $\frac{d^k S_n}{dx^k}(0) = c\binom{n+k}{k}k!\,L_n^{(k-1)}(0) = 0$

Which implies $c = 0$ as $L_n^{(k-1)}(0) \neq 0$ hence $S_n(x) \equiv 0$ which completes the proof of the uniqueness.

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