Formation of Abelian Group by Mobius (α,β) Function

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Abstract: ***In this paper we have generated the algorithm to find the value of Mobius function for any positive number n and define some properties of Mobius (α,β) function. This paper also aims to show that the Mobius (α,β) function satisfies the properties of an abelian group.***

Keywords: Mobius function, Algorithm, Mobius (α,β) function, Abelian group.

1. Introduction

There exist many number theoretic functions, which includes Divisor function $\tau(n)$ [1], Sigma function $\sigma(n)$ [1], Euler phi function $\upsilon(n)$ [2] and Mobius function $\mu(n)$ [2]. All these functions play very important role in the field of number theory [3]. Mobius function is a special number theoretic function which satisfies some special properties and is useful in the formation of group in the field of abstract algebra [4].

2. Mobius Function

The Mobius function $\mu(n)$ is defined for all positive integers n and has its values in {−1, 0, 1} depending on the factorization of n into prime factors. It is defined as follows:

1, if n is a square-free positive integer with an even number of prime factors.

$$\mu(n) = \begin{cases} 1, \text{ if } n \text{ is a square-free positive integer with an even number of prime factors.} \\ -1, \text{ if } n \text{ is a square-free positive integer with an odd number of prime factors.} \\ 0, \text{ if } n \text{ has a squared prime factor.} \end{cases}$$

2.1 Program of Mobius Function

```
import java.io.*;
classmobius
{
  public static void main(String args[]) throws IOException
  {
    InputStreamReader reader=new
    InputStreamReader(System.in);
    BufferedReader input=new BufferedReader(reader);

    intn,i,j,c=0,p;
    boolean flag=true;
    int a[]=new int[100];

    System.out.println("Enter any number....");
    n=Integer.parseInt(input.readLine());
    p=n;

    if(n==1)
    System.out.println("mobius(1)=1");
    else
    {
      do
      {
        for(i=2;i<=n;i++)
        {
          if(n%i==0)
          {
            break;
          }
        }
        a[c++]=i;
        n=n/i;
      }
      while(n>1);

        for(i=0;i<c-1;i++)
        {
          for(j=i+1;j<c;j++)
          {
            if(a[i]==a[j])
            {
              flag=false;
              break;
            }
          }
        }
        if(flag==true)
        {
          int r;
          r=(int)Math.pow(-1,c);
          System.out.println("mobius("+p+")="+r);
        }
        else
        {
          System.out.println("mobius("+p+")="+0);
        }
    }
  }
}
```

3. Mobius (α,β) Function

The function [μ(α,β);n] is known as the Mobius (α,β) function for all positive numbers n, having value $(-1)^k$ and 0, depends upon prime factors of n lies in the interval [α,β]. The function [μ(α,β);n] is defined as follows:

$$[\mu(\alpha,\beta);n] = \begin{cases} (-1)^k, \text{ if all } p_i \text{ are distinct and lies in the interval } [\alpha,\beta]. \\ 0\ , \text{ if } p_i^2 | n \text{ and } p_i \in [\alpha,\beta]. \end{cases}$$

where, α, β ϵ N and k is the number of distinct prime factors of n lies in the interval [α,β] and [μ(1,β);1]= 1

eg: $[\mu(3,12);195] = [\mu(3,12);3.5.13] = (-1)^2 = 1$

3.1 Properties of Mobius (α,β) Function

3.1.1 [μ(α,β);n] = μ(n), for every n ϵ N, if all the prime factors of n are distinct and lies in the interval [α,β].

3.1.2 [μ(α,β);n] is multiplicative, for every n ϵ N, if all the prime factors of n are distinct and lies in the interval [α,β].
i.e.,$[\mu(\alpha,\beta);n_1].[\mu(\alpha,\beta);n_2]....[\mu(\alpha,\beta);n_r] = [\mu(\alpha,\beta);n_1.n_2.....n_r]$
if gcd $(n_1,n_2,\ldots,n_r) = 1$

Proof: To prove this property first we consider only two distinct numbers n_1 and n_2 such that, gcd $(n_1,n_2) = 1$

Let $n_1 = p_1.\ p_2.....\ p_i$ has i distinct prime factors, all lies in the interval [α,β],
and $n_2 = q_1.\ q_2.....\ q_j$ has j distinct prime factors, all lies in the interval [α,β].

then $n_1.\ n_2 = p_1.\ p_2.....\ p_i.\ q_1.\ q_2.....\ q_j$ has (i+j) distinct prime factors, all lies in the interval [α,β].

Therefore,
$[\mu(\alpha,\beta);n_1].\ [\mu(\alpha,\beta);n_2] = (-1)^i.\ (-1)^j$
$= (-1)^{i+j}$
$= [\mu(\alpha,\beta);p_1.\ p_2.....\ p_i.\ q_1.\ q_2.....\ q_j]$
$= [\mu(\alpha,\beta);n_1.\ n_2]$
where, gcd $(n_1,n_2) = 1$

In general, we can write
if gcd $(n_1,n_2,\ldots,n_r) = 1$

then,
$[\mu(\alpha,\beta);n_1].[\mu(\alpha,\beta);n_2].....[\mu(\alpha,\beta);n_r] = [\mu(\alpha,\beta);n_1.n_2....n_r]$.

Hence, [μ(α,β);n] is multiplicative.

3.1.3 ([μ(α,β);n], *) is an abelian group, for every n ϵ N, if all the prime factors of n are distinct and lies in the interval [α,β].
i.e., [μ(α,β);n] with a binary operation multiplication (*) form an abelian group, where, α,β ϵ N, n ϵ Z^+ and all prime factors of n lies in the interval [α,β].

Proof: Let S = [μ(α,β);n], where all the prime factors of n are distinct and lies in the interval [α,β].
then, S = {-1, 1} and multiplication (*) be the binary operation on the set S.

Closure and Associative Property:

Composition table for S for the binary operation (*) is as follows:

Table 1: Composition table

*	-1	1
-1	1	-1
1	-1	1

Since all the elements of composition table belongs to the set S, therefore closure and associative property hold.

Existence of identity:

Since, (-1)*(1) = -1 = (1)*(-1)
and (1)*(1) = 1 = (1)*(1)
i.e., for every s ϵ S, there exist 1 ϵ S, such that
(s)*(1) = s = (1)*(s)

Hence, 1 is the identity element for multiplication.

Existence of inverse:

Since, (-1)*(-1) = 1 = (-1)*(-1)
and (1)*(1) = 1 = (1)*(1)
i.e., for every s ϵ S, there exist s^{-1} ϵ S, such that
$(s)*(s^{-1}) = 1 = (s^{-1})*(s)$

therefore, each element s ϵ S has its multiplicative inverse in S.
Hence, ([μ(α,β);n], *) is a group.

Also, (1)*(-1) = -1 = (-1)*(1)
(1)*(1) = 1 = (1)*(1)
(-1)*(-1) = 1 = (-1)*(-1)
Therefore the commutative law also holds.

Hence, ([μ(α,β);n], *) is an abelian group.

4. Conclusion

The above analysis shows that the function [μ(α,β);n] satisfy the multiplicative law and all the properties of an ablelian group together with the binary operation (*) multiplication. Hence the function [μ(α,β);n] forms an abelian group.

References

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