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Left-Invertible Matrices

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Abstract: ***In this paper we consider general rectangular matrices over the given field F, and we characterize the class of all left inverses of the given rectangular matrix provided that it is left invertible.***

Keywords: Injective matrix; left-cancellable matrix; full-column rank; a basis for a vectorspace.

1. Introduction

A square matrix A of order n over the field F, is said to be an invertible, if there exist another matrix B of order n over the given field F, such that $BA = I_n = AB$, where I_n is an identity matrix of order n. If such B exists then it is unique and it is called the inverse of A. It is known that a given square matrix is invertible if and only if it has a full column rank as well as a full row rank, otherwise it is non invertible. More generally, non-square matrices are noninvertible in the above sense. However, there are non-square matrices having an inverse from one side only; from the left side (respectively from the right side) and hence we call them left invertible (respectively right invertible matrices). It is a fact that, an $m \times n$ rectangular matrix has an inverse from the left side (respectively from the right side) if and only if it has a rank n (respectively m). Unfortunately, if the left side (respectively the right side) inverse of a rectangular matrix exists, then it needs not to be necessarily unique. The question in this case is that how many left (respectively right) inverses can be there for a given rectangular matrix? In this paper we treat only those left invertible matrices and we characterize what the set of all left inverses of a rectangular matrix looks like.

2. Preliminaries

Definition 2.1: Let F be a field and let A be an $m \times n$ matrix over F. Then A is called an injection if:

$$Ax = Ay \Rightarrow x = y \text{ for all } x, y \in F^n.$$

Definition 2.2: For an $m \times n$ matrix A over a field F, define the null space of A and the column space of A respectively by:

$$\mathcal{N}(A) = \{x \in F^n : Ax = 0\}$$
$$\mathcal{R}(A) = \{Ax : x \in F^n\}$$

Then both $\mathcal{N}(A)$ and $\mathcal{R}(A)$are subspaces of F^n and F^m respectively. [1]

Theorem 2.1: For an $m \times n$ matrix A over a field F:

$$\dim(\mathcal{N}(A)) + \dim(\mathcal{R}(A)) = \dim F^n = n \ [1]$$

Theorem 2.2: An $m \times n$ matrix A over a field F is an injection if and only if $\mathcal{N}(A) = \{0\}$

Proof: Suppose that A is an injection. Then it is clear that $A0 = 0 \Longrightarrow 0 \in \mathcal{N}(A) \Longrightarrow \{0\} \subseteq \mathcal{N}(A)$.

On the other hand:

$$x \in \mathcal{N}(A) \Longrightarrow Ax = 0 = A0$$
$$\Longrightarrow Ax = A0$$
$$\Longrightarrow x = 0 \quad (\because A \text{ is an injection})$$
$$\Longrightarrow \mathcal{N}(A) \subseteq \{0\} \subseteq \mathcal{N}(A)$$

Therefore, $\mathcal{N}(A) = \{0\}$.

Conversely suppose that $\mathcal{N}(A) = \{0\}$. Therefore for any $x, y \in F^n$; $Ax = Ay \Longrightarrow A(x - y) = 0 \Longrightarrow x - y \in \mathcal{N}(A) = \{0\} \Longrightarrow x - y = 0 \Longrightarrow x = y$ and hence A is an injection; ■

Theorem 2.3: If an $m \times n$ matrix A over a field F is an injection, then $n \leq m$.

Proof: Suppose that A is an injection. Thus it follows from the above theorem that $\mathcal{N}(A) = \{0\}$ and hence $\dim(\mathcal{N}(A)) = 0$. Therefore;

$$n = \dim(\mathcal{N}(A)) + \dim(\mathcal{R}(A))$$
$$\Longrightarrow n = 0 + \dim(\mathcal{R}(A))$$
$$\Longrightarrow n = \dim(\mathcal{R}(A)) \leq \dim F^m = m.$$
$$\Longrightarrow n \leq m. \ ■$$

Theorem 2.4: If an $m \times n$ matrix A over a field F is an injection, then its columns $A_1, A_2, \ldots, A_n$ are linearly independent vectors in the vectorspace F^m.

Proof: Suppose that A is an injection. Then by theorem 2 we have that, $\mathcal{N}(A) = \{0\}$

Now for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ in F;

$\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n = 0 \Longrightarrow Ax = 0$ where $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ in F^n

$$\Longrightarrow x \in \mathcal{N}(A) = \{0\}$$
$$\Longrightarrow x = 0$$
$$\Longrightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$$

Thus $A_1, A_2, \ldots, A_n$ are linearly independent to each other.

Conversely suppose that the columns $A_1, A_2, \ldots, A_n$ of A are linearly independent to each other. Following the above theorem, it suffices to prove that $\mathcal{N}(A) = \{0\}$. Then it is clear that, $A0 = 0 \Longrightarrow 0 \in \mathcal{N}(A) \Longrightarrow \{0\} \subseteq \mathcal{N}(A)$.

On the other hand; for any $x = (x_1, x_2, \ldots, x_n)^T$ in F^n consider:

$$x \in \mathcal{N}(A) \Longrightarrow Ax = 0$$
$$\Longrightarrow x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = 0$$
$$\Longrightarrow x_1 = 0, x_2 = 0, \ldots x_n = 0$$

$(\because A_1, A_2, \ldots, A_n$ are linearly independent)

$$\Longrightarrow x = 0$$
$$\Longrightarrow \mathcal{N}(A) \subseteq \{0\} \subseteq \mathcal{N}(A)$$
$$\Longrightarrow \mathcal{N}(A) = \{0\} \ ■$$

3. Left-Invertible Matrices

Definition 3.1: An $m \times n$ matrix A over a field F is said to be left invertible if there exists an $n \times m$ matrix B over the given field F such that $BA = I_n$ where I_n is an identity matrix of order n. in this case B is called the left inverse of A

Definition 3.2: An $m \times n$ matrix A over a field F is said to be left cancellable if for any positive integer p and for any two $n \times p$ matrices B and C over the over the given field F;

$$AB = AC \Longrightarrow B = C.$$

The next theorem gives us two equivalent conditions (necessary and sufficient conditions) to a given rectangular matrix to have a left inverse and it is included here for the completeness of the paper.

Theorem 3.1: The following are equivalent for any $m \times n$ matrix A over the given field F.

(1) A is an injection
(2) A is Left Invertible
(3) A is Left cancellable

Proof: (1) $\Longrightarrow$ (2) Suppose that A is an injection. Then the columns $A_1, A_2, \ldots, A_n$ of A are linearly independent in F^m and hence forms a basis for the column space $\mathcal{R}(A)$ of A in F^m so that $\dim(\mathcal{R}(A)) = n$. Put $y_1 = A_1$, $y_2 = A_2$, …,$y_n = A_n$, then $\{y_1, y_2, \ldots, y_n\}$ is a basis for $\mathcal{R}(A)$. If $n = m$, then A is a square matrix having a full column rank and hence invertible. Assume that $n < m$ and let $r = m - n > 0$. Therefore $\mathcal{R}(A)$ is a proper subspace of F^m spanned by the column vectors $y_1, y_2, \ldots, y_n$ of A. Thus we can choose an element y_{n+1} in $F^m - \mathcal{R}(A)$. Since $y_{n+1} \notin \mathcal{R}(A)$ then it is not a scalar combination of those y_i's and hence the set $\{y_1, y_2, \ldots, y_n, y_{n+1}\}$ becomes linearly independent in F^m. Let U_1 be the subspace of F^m generated by $\{y_1, y_2, \ldots, y_n, y_{n+1}\}$. Therefore since $\{y_1, y_2, \ldots, y_n, y_{n+1}\}$ is linearly independent we get that $\dim U_1 = n + 1$. If $n + 1 = m$ then $U_1 = F^m$ and if $n + 1 < m$, then we can choose another element y_{n+2} in $F^m - U_1$, then $y_{n+2} \notin U_1$, so that this y_{n+2} is not a linear combination of vectors $y_1, y_2, \ldots, y_n, y_{n+1}$ and hence the set $\{y_1, y_2, \ldots, y_n, y_{n+1}, y_{n+2}\}$ is linearly independent in F^m. Similarly doing this process r times, we get r vectors $y_{n+1}, y_{n+2}, \ldots, y_{n+r} = y_m$ in F^m such that the set $B = \{y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_{n+r} = y_m\}$ is alinerly independent in F^m and hence an ordered basis for F^m. If we let $A[B]$ to be the matrix with columns $y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_m$; vectors in B, then $A[B]$ is an $m \times m$ matrix such that its first n columns are columns of A. On the other hand, since its columns $y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_m$ are linearly independent we get that $A[B]$ is invertible and let $A[B]^{-1}$ be its inverse. Then this $A[B]^{-1}$ is an $m \times m$ matrix over the same field F such that $A[B]^{-1}A[B] = I_m = A[B]A[B]^{-1}$; that is, $A[B]^{-1}y_i = f_i$ for all $1 \le i \le m$ and in particular for $1 \le i \le n$, where $f_i = (0, 0, \ldots, 0, 1, 0, \ldots 0)$; an m tuple of 0's and 1 at the i^{th} place. In other words, f_i is the i^{th} column of the identity matrix I_m. Now define an $n \times m$ matrix denoted by A_B, such that its rows are the first n rows of $A[B]^{-1}$. For each $1 \le i \le n$ and $1 \le j \le n$, if we let x_{ij} to be an entry of the product matrix $A_B A$ which is placed at the i^{th} row and at the j^{th} column, then

x_{ij} = the dot product of the i^{th} row of A_B and the j^{th} column of A

= the dot product of the i^{th} row of $A[B]^{-1}$ and the j^{th} column of $A[B]$

$$= \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases} \qquad (\because A[B]^{-1}A[B] = I_m)$$

Therefore $A_B A = I_n$ and hence this A_B is a left inverse of A. Thus A is left invertible.

(2) $\Longrightarrow$ (3) Suppose that A is left invertible. Then it has a left inverse and let B be a left inverse of A. Then B is an $n \times m$ matrix over the same field F such that $BA = I_n$. For any integer p, let C and D be an $n \times p$ matrices over the given field F such that $AC = AD$. Therefore $B(AC) = B(AD) \Longrightarrow (BA)C = (BA)D \Longrightarrow I_n C = I_n D \Longrightarrow C = D$ and hence A is left cancellable.

(3) $\Longrightarrow$ (1) is trivial.

Remark: We observe from the above theorem that any injective matrix has at least one left inverse and in fact it is not necessarily unique. So, the question in this case is that how many left inverses can be there for a given injective matrix? In the next theorem we characterize the set of all left inverses of a given injective matrix.

Theorem 3.2: Let A be an $m \times n$ injective matrix over the field F. If we let

$\mathfrak{L}(A)$ = be the class of all left inverses of A, and let
$\mathfrak{B}(A) = \{B : B\ is\ a\ basis\ for\ F^m$, containing the columns $A_1, A_2, \ldots, A_n$ of A and $span[B_1 - \{A_1, A_2, \ldots, A_n\}] \cap span[B_2 - \{A_1, A_2, \ldots, A_n\}] = \{0\}$ for any $B_1 \neq B_2 \in \mathfrak{B}(A)\}$
Therefore there is a one-to-one correspondence between $\mathfrak{L}(A)$ and $\mathfrak{B}(A)$.

Proof: If $B = \{y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_m\} \in \mathfrak{B}(A)$, then B is a basis for F^m containing the columns $A_1, A_2, \ldots, A_n$ of A and by simple rearrangement we can assume that $y_i = A_i$ for $1 \le i \le n$. For each $B \in \mathfrak{B}(A)$ let us define two matrices $A[B]$ and A_B as in the above theorem (Theorem 3.1). Thus it follows that $A_B \in \mathfrak{L}(A)$.
Now define $h: \mathfrak{B}(A) \to \mathfrak{L}(A)$ by:

$$h(B) = A_B \text{ for all } B \in \mathfrak{B}(A)$$

It is clearly observed in theorem 3.1 that, this h is well defined. Now we first prove that h is one-to-one.
Let $B_1 = \{y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_m\}$ and $B_2 = \{z_1, z_2, \ldots, z_n, z_{n+1}, \ldots, z_m\} \in \mathfrak{B}(A)$ such that $B_1 \neq B_2$. Therefore $y_i = z_i = A_i$ for all $1 \le i \le n$ and $span[B_1 - \{A_1, A_2, \ldots, A_n\}] \cap span[B_2 - \{A_1, A_2, \ldots, A_n\}] = \{0\}$; that is $span\{y_{n+1}, \ldots, y_m\} \cap span\{z_{n+1}, \ldots, z_m\} = \{0\}$. If $A[B_1]$ and $A[B_2]$ are matrices with columns $y_1, y_2, \ldots, y_m$ and $z_1, z_2, \ldots, z_m$ respectively, then $A[B_1]$ and $A[B_2]$ are both invertible. Let A_{B_1} and A_{B_2} be an $n \times m$ matrices containing only the first n rows of $A[B_1]^{-1}$ and $A[B_2]^{-1}$ respectively. As it is observed in the above theorem, both A_{B_1} and A_{B_2} are left inverses of A, so that $A_{B_1}A = I_n = A_{B_2}A$, in other words, for $1 \le i \le n$, $A_{B_1}y_i = A_{B_1}A_i = e_i$ and similarly $A_{B_2}y_i = A_{B_2}A_i = e_i$ where e_i is the i^{th} column of the identity matrix I_n. Also for $1 \le j \le m - n$, $A_{B_1}y_{n+j} = 0 = A_{B_2}z_{n+j}$. Now our claim is to prove that $A_{B_1} \neq A_{B_2}$. Here we use a proof by

contradiction and suppose if possible that $A_{B_1} = A_{B_2}$, then $A_{B_1}y = A_{B_2}y$ for all $y \in F^m$. Now choose exactly one element $y \in F^m - \mathcal{R}(A)$, then $y \notin \mathcal{R}(A)$ and hence y is not a linear combination of the columns $A_1, A_2, \dots, A_n$ of A. Considering a basis B_1, y can be expressed as $y = \alpha_1 y_1 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m = \alpha_1 A_1 + \cdots + \alpha_n A_n + \alpha_{n+1} y_{n+1} + \cdots + \alpha_m y_m$ for some scalars $\alpha_1, \alpha_2, \dots, \alpha_m$. On the other hand, considering a basis B_2, y can also be expressed as $y = \beta_1 z_1 + \cdots + \beta_n z_n + \cdots + \beta_m z_m = \beta_1 A_1 + \cdots + \beta_n A_n + \beta_{n+1} z_{n+1} + \cdots + \beta_m z_m$ for some scalars $\beta_1, \beta_2, \dots \beta_m$. Therefore,

$A_{B_1}y = A_{B_2}y$

$\Longrightarrow A_{B_1}(\alpha_1 A_1 + \cdots + \alpha_n A_n + \alpha_{n+1} y_{n+1} + \cdots + \alpha_m y_m) = A_{B_2}(\beta_1 A_1 + \cdots + \beta_n A_n + \beta_{n+1} z_{n+1} + \cdots + \beta_m z_m)$

$\Longrightarrow \alpha_1 A_{B_1} A_1 + \cdots + \alpha_n A_{B_1} A_n + \alpha_{n+1} A_{B_1} y_{n+1} \dots + \alpha_m A_{B_1} y_m = \beta_1 A_{B_2} A_1 + \cdots + \beta_n A_{B_2} A_n + \beta_{n+1} A_{B_2} z_{n+1} + \cdots + \beta_m A_{B_2} z_m$

$\Longrightarrow \alpha_1 e_1 + \cdots + \alpha_n e_n + \alpha_{n+1} 0 + \cdots + \alpha_m 0 = \beta_1 e_1 + \cdots + \beta_n e_n + \beta_{n+1} 0 + \cdots + \beta_m 0$

$\Longrightarrow \alpha_1 e_1 + \cdots + \alpha_n e_n = \beta_1 e_1 + \cdots + \beta_n e_n$

$\Longrightarrow (\alpha_1 - \beta_1) e_1 + \cdots + (\alpha_n - \beta_n) e_n = 0$

$\Longrightarrow \alpha_1 - \beta_1 = 0, \alpha_2 - \beta_2 = 0, \dots, \alpha_n - \beta_n = 0$

($\because \{e_1, \dots, e_n\}$ is a standard basis for F^n)

$\Longrightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_n = \beta_n$

Since $\alpha_1 A_1 + \cdots + \alpha_n A_n + \alpha_{n+1} y_{n+1} + \cdots + \alpha_m y_m$ and $\beta_1 A_1 + \cdots + \beta_n A_n + \beta_{n+1} z_{n+1} + \cdots + \beta_m z_m$ are different forms of y, then

$\alpha_1 A_1 + \cdots + \alpha_n A_n + \alpha_{n+1} y_{n+1} + \cdots + \alpha_m y_m = \beta_1 A_1 + \cdots + \beta_n A_n + \beta_{n+1} z_{n+1} + \cdots + \beta_m z_m$

$\Longrightarrow \alpha_1 A_1 + \cdots + \alpha_n A_n + \alpha_{n+1} y_{n+1} + \cdots + \alpha_m y_m = \alpha_1 A_1 + \cdots + \alpha_n A_n + \beta_{n+1} z_{n+1} + \cdots + \beta_m z_m$

$\Longrightarrow \alpha_{n+1} y_{n+1} + \alpha_{n+2} y_{n+2} + \cdots + \alpha_m y_m = \beta_{n+1} z_{n+1} + \beta_{n+2} z_{n+2} + \cdots + \beta_m z_m$

If we let $u = \alpha_{n+1} y_{n+1} + \cdots + \alpha_m y_m = \beta_{n+1} z_{n+1} + \cdots + \beta_m z_m$, then u is spanned by $\{y_{n+1}, \dots, y_m\}$ aswellas $\{z_{n+1}, \dots, z_m\}$ and hence $u \in span\{y_{n+1}, \dots, y_m\} \cap span\{z_{n+1}, \dots, z_m\} = \text{span}[B_1 - \{A_1, \dots, A_n\}] \cap \text{span}[B_2 - \{A_1, \dots, A_n\}] = \{0\}$. So that $u = 0$. Thus we have that: $u = \alpha_{n+1} y_{n+1} + \cdots + \alpha_m y_m = 0$ and $\beta_{n+1} z_{n+1} + \cdots + \beta_m z_m = 0$. Since each y_i's and z_i's are linearly independent respectively, it follows that $\alpha_{n+j} = 0 = \beta_{n+j}$ for all $1 \le j \le m - n$. Therefore, $y = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n$ and hence $y \in \mathcal{R}(A)$ which is a contradiction to our choice of y. Thus $A_{B_1} \neq A_{B_2}$. This says that $h(B_1) \neq h(B_2)$ for all $B_1 \neq B_2 \in \mathfrak{B}(A)$. Therefore h is a one to one map.

Furthermore, we prove that h is an on-to map; for,

Let $D \in \mathfrak{L}(A)$, then D is a left inverse of A; that is, D is an $n \times m$ matrix over the given field F such that $DA = I_n$ (an identity matrix of order n). So that $D(Ax) = x$ for all $x \in F^n$. In particular, $D(Ae_i) = e_i$ and hence $DA_i = e_i$ for all $1 \le i \le n$, where e_i is the i^{th} column of the identity matrix I_n.

$x \in F^n \Longrightarrow x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$ for some scalars $\alpha_1, \alpha_2, \dots, \alpha_n$

$\Longrightarrow x = \alpha_1 D(Ae_1) + \alpha_2 D(Ae_2) + \cdots + \alpha_n D(Ae_n)$

$\Longrightarrow x = D(\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n)$

$\Longrightarrow x = Dy$, where $y = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n \in F^m$

$\Longrightarrow x \in \mathcal{R}(D)$

$\Longrightarrow F^n \subseteq \mathcal{R}(D) \subseteq F^n$

$\Longrightarrow \mathcal{R}(D) = F^n$ and hence $\dim(\mathcal{R}(D)) = \dim F^n = n$.

Since D is an $n \times m$ matrix over the field F, we have that:

$$\dim(\mathcal{N}(D)) + \dim(\mathcal{R}(D)) = \dim F^m$$
$$\Longrightarrow \dim(\mathcal{N}(D)) + n = m$$

In this case if $m \neq n$, then $\dim(\mathcal{N}(D)) = m - n > 0$, and hence $\mathcal{N}(D)$ is a nontrivial subspace of F^m with dimension $m - n$. So that we can choose $m - n$ linearly independent vectors $A_{n+1}, A_{n+2}, \dots, A_m$ in $\mathcal{N}(D)$. Thus the product $DA_{n+j} = 0$ for all $1 \le j \le m - n$.

Claim 1: The set $B = \{A_1, A_2, \dots, A_n, A_{n+1}, \dots, A_m\}$ is linearly independent in F^m.

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m$;

$$\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n + \cdots + \alpha_m A_m = 0 \quad (1)$$

$\Longrightarrow D(\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n + \cdots + \alpha_m A_m) = 0$

$\Longrightarrow \alpha_1 DA_1 + \alpha_2 DA_2 + \cdots + \alpha_n DA_n + \alpha_{n+1} DA_{n+1} \dots + \alpha_m DA_m = 0$

$\Longrightarrow \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$ ($\because DA_i = e_i$ for all $1 \le i \le n$ and $DA_{n+j} = 0$ for all $1 < j \le m - n$)

$\Longrightarrow \alpha_1 = 0,\ \alpha_2 = 0, \dots, \alpha_n = 0.$

($\because \{e_1, \dots, e_n\}$ is a standard basis for F^n)

Substituting the value of these α_i's in equation (1) we get that: $\alpha_{n+1} A_{n+1} + \cdots + \alpha_m A_m = 0$ and since each A_{n+j}'s are linearly independent to each other, it follows that $\alpha_{n+j} = 0$ for all $1 \le j \le m - n$. This says that, $\alpha_i = 0$ for all $1 \le i \le m$. Therefore the set $B = \{A_1, A_2, \dots, A_n, A_{n+1}, \dots, A_m\}$ is linearly independent in F^m and since B has exactly m elements, then it becomes a basis for F^m containing the columns $A_1, A_2, \dots, A_n$ of A, so that $B \in \mathfrak{B}(A)$.

Claim 2: $D = A_B = h(B)$

Since B forms a basis for F^m, any $y \in F^m$ can be expressed as $y = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n + \cdots + \alpha_m A_m$, then

$Dy = D(\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n + \cdots + \alpha_m A_m)$

$= \alpha_1 DA_1 + \alpha_2 DA_2 + \cdots + \alpha_n DA_n + \cdots + \alpha_m DA_m$

$= \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$ ($\because DA_i = e_i$ for all $1 \le i \le n$ and $DA_{n+j} = 0$ for all $1 < j \le m - n$)

$= A_B(\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n + \cdots + \alpha_m A_m)$

$= A_B y$

Thus $Dy = A_B y$ for all $y \in F^m$, and this implies that $D = A_B = h(B)$; that is, h is an on-to map and hence a one-to-one correspondence. ■

4. Conclusion

As we observe in Theorem 2.2, Theorem 2.4 and Theorem 3.1 of this paper, a given rectangular $m \times n$ matrix A over the given field F, has a left inverse if and only if it is an injection or equivalently if it is left cancellable or equivalently if $\mathcal{N}(A) = \{0\}$ or equivalently if its columns are linearly independent or equivalently if it has a full column rank. This tells us that any rectangular matrix satisfying one of the above conditions and hence the others, has at least one left inverse. If A is square, then the existing left inverse is also a right inverse and hence it is unique. Otherwise, it needs not to be necessarily unique. As proved in Theorem 3.2, there is a one to one correspondence between the set $\mathfrak{L}(A)$ of all left inverses of A and the set $\mathfrak{B}(A)$ of bases for F^m, containing the columns $A_1, A_2, \dots, A_n$ of A such that, $span[B_1 - \{A_1, A_2, \dots, A_n\}] \cap span[B_2 - \{A_1, A_2, \dots, A_n\}] =$

$\{0\}$ for all $B_1 \neq B_2 \in \mathfrak{B}(A)$. On other words, the set of all left inverses of an $m \times n$ matrix A over the field F, is equivalent (or cardinal) with the collection of tuples of $m - n$ column vectors $A_{n+1}, A_{n+2}, \ldots, A_m$ in F^m satisfying that the set $\{A_1, A_2, \ldots, A_n, A_{n+1}, A_{n+2}, \ldots, A_m\}$ is linearly independent and vectorspaces U_1 and U_2 generated by any two arbitrary tuples $A_{n+1}, A_{n+2}, \ldots, A_m$ and $A'_{n+1}, A'_{n+2}, \ldots, A'_m$ respectively, have only a zero vector in common.

References

[1] Kwak, J. H., Hong, S., Linear Algebra, Birkhauser, Boston, 1948.
[2] Nicholson, W.K, Linear Algebra with Applications, 3rd Edition, PWS Publishing Hous, Boston, 1990
[3] Shrma, A.K., Prakash, M., *Linear Transformation*, Discovery Publishing House, 2007. Page 1-104.
[4] Smith, L., Linear Algebra: *Undergraduate Texts in Mathematics,* Springer Science & Business Media, 2012. Page 62-99
[5] Szabo, F., Linear Algebra: *An Introduction Using Mathematica*, Academic Press, 2000. Page 321-374.