

# Four Component Electromagnetic Field Equations in Tensor Form

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**Abstract:** Dirac-Maxwell's equations are generalized by introducing electric scalar field and magnetic scalar field. Such attempt removes Lorentz condition on the field potentials and allows the electric as well as magnetic charges to vary with time. Therefore, the charge on a single particle is not necessary to be conserved. Any loss in charge causes to raise charge on another particle in order to conserve the net charge once again. The increase or decrease in charge on the particle causes to produce a scalar field which makes effort to decrease or increase charge on another particle maintaining the net charge constant in the space. Dirac-Maxwell's equations are generalized in order to satisfy such fields. Further these equations are written in tensor form.

**Keywords:** Generalized Dirac-Maxwell's equations, Lorentz Gauge, electric scalar field, magnetic scalar field, Tensor

## 1. Introduction

In Maxwell's equations, the symmetry is obtained by Dirac [1-4] by introducing magnetic monopoles as the source of static magnetic field. When any magnetic monopole comes into a motion, induces or produces another electric field. Using magnetic monopoles as source of magnetic field, Dirac generalized the Maxwell's equations, called as Dirac-Maxwell's equations (DME), which are

$$\nabla \cdot \mathbf{E} = 4\pi\rho^e \quad (1a)$$

$$\nabla \cdot \mathbf{H} = 4\pi\rho^m \quad (1b)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} = -\frac{4\pi}{c} \mathbf{j}^m \quad (1c)$$

$$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{j}^e \quad (1d)$$

where  $\rho^e$  and  $\rho^m$  are the electric charge and magnetic monopole densities respectively, and  $\mathbf{j}^e$  and  $\mathbf{j}^m$  are electric and magnetic current densities respectively. Solutions of these DME are by Cabibbo & Ferrari [5], Epstein [6], Ferrari [7] as

$$\mathbf{E} = -\nabla\phi^e - \frac{1}{c} \frac{\partial \mathbf{A}^e}{\partial t} - \nabla \times \mathbf{A}^m \quad (2a)$$

$$\mathbf{H} = -\nabla\phi^m - \frac{1}{c} \frac{\partial \mathbf{A}^m}{\partial t} + \nabla \times \mathbf{A}^e \quad (2b)$$

The sources are subjected to the continuity equation and conserved.

$$\nabla \cdot \mathbf{j}^{e,m} + \frac{\partial \rho^{e,m}}{\partial t} = 0 \quad (3a)$$

This condition on the sources then leads to the Lorentz gauge on the four potentials, viz.

$$\nabla \cdot \mathbf{A}^{e,m} + \frac{1}{c} \frac{\partial \phi^{e,m}}{\partial t} = 0 \quad (3b)$$

Thus the Lorentz condition on the potentials makes the electric charges and magnetic monopoles time independent. In order to make the charges time dependent, these DME are generalized [8-10] by introducing electric scalar field  $E_0$  and magnetic scalar field  $H_0$  which become replacement of the Lorentz gauges on the electric and magnetic potentials. These Generalized Dirac-Maxwell's equations (GDME) are

$$\nabla \cdot \mathbf{E} = \frac{1}{c} \frac{\partial E_0}{\partial t} + 4\pi\rho^e \quad (4a)$$

$$\nabla \cdot \mathbf{H} = \frac{1}{c} \frac{\partial H_0}{\partial t} + 4\pi\rho^m \quad (4b)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} + \nabla H_0 = -\frac{4\pi}{c} \mathbf{j}^m \quad (4c)$$

$$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \nabla E_0 = \frac{4\pi}{c} \mathbf{j}^e \quad (4d)$$

These have usual solutions for  $\mathbf{E}$  and  $\mathbf{H}$  given by equations (2) and in addition to that they have solutions for scalar fields as

$$E_0 = \nabla \cdot \mathbf{A}^e + \frac{1}{c} \frac{\partial \phi^e}{\partial t} \quad (5a)$$

$$H_0 = \nabla \cdot \mathbf{A}^m + \frac{1}{c} \frac{\partial \phi^m}{\partial t} \quad (5b)$$

where  $E_0$  and  $H_0$  are the electric and magnetic scalar fields respectively. Clearly these scalar fields are the removal of the Lorentz gauge on their respective potentials indicating the continuity equation need not be hold by the sources but again total amount of charge should be conserved. This happens such that the decrease in charge on any particle will cause to produce its kind of scalar field which links to another particle and causes to increase charge on this particle in order to compensate the loss of charge on the first particle. Therefore, the scalar field generated in this process should be proportional to the rate of change of charge on the particle. If the charge becomes time-independent then the scalar field vanishes with satisfying the Lorentz condition by the potentials and the GDME then reduce to the original DME. Hence the set of GDME given by equations (4) becomes proper generalization of the Dirac-Maxwell equations.

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## 2. GDME in tensor form - electric charges as source of fields

To write GDME in tensor form in Minkowski space we first consider them with source of electric type only as

$$\nabla \cdot \mathbf{E} = \frac{1}{c} \frac{\partial E_0}{\partial t} + 4\pi\rho \quad (6a)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (6b)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} = 0 \quad (6c)$$

$$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \nabla E_0 = \frac{4\pi}{c} \mathbf{j} \quad (6d)$$

The solutions are

$$\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \quad (7a)$$

$$\mathbf{H} = \nabla \times \mathbf{A} \quad (7b)$$

$$E_0 = \nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \quad (7c)$$

The four current densities and the four potentials have the forms as

$$j_\mu = (\mathbf{j}, ic\rho) \quad (8a)$$

$$A_\mu = (\mathbf{A}, i\phi) \quad (8b)$$

Similarly the four space-time coordinates and their derivatives are as

$$x_\mu = (\nabla, ict) \quad (9a)$$

$$\delta_\mu = \left( \nabla, -\frac{1}{c} \frac{\partial}{\partial t} \right) \quad (9b)$$

The generalized form of the electromagnetic field tensor is then

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} - \delta_{\mu\nu} \delta_{\alpha\beta} \frac{\partial A_\beta}{\partial x_\alpha} \quad (10)$$

In this equation the last term is additional one used for the scalar field  $E_0$ . Calculating all the components of the field tensor, we obtain

$$F_{\mu\nu} = \begin{Bmatrix} -E_0 & H_z & -H_y & -iE_x \\ -H_z & -E_0 & H_x & -iE_y \\ H_y & -H_x & -E_0 & -iE_z \\ iE_x & iE_y & iE_z & -E_0 \end{Bmatrix} \quad (11)$$

The generalized form of the electromagnetic field tensor is not antisymmetric as its trace is non zero but it antisymmetric with respect to the non-diagonal elements.

By using the above field tensor, the GDME (6) can be expressed as usual. Consider the following tensor equation

$$\frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0 \quad (12)$$

If we choose  $\lambda, \mu, \nu$  to be any combination of 1, 2, 3 then this equation reduces to

$$\frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} = 0 \quad (13)$$

$$\frac{\partial H_{12}}{\partial x_3} + \frac{\partial H_{23}}{\partial x_1} + \frac{\partial H_{31}}{\partial x_2} = 0 \quad (14)$$

This is equation (6b) of GDME.

Similarly, if we set one of the indices  $\lambda, \mu, \nu$  equal to 4, it represents equation (6c).

The other two homogeneous generalized Maxwell's equations may be obtained from

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} j_\mu \quad (15)$$

For  $\mu = 1, 2, 3$  it represents equation (6d). For  $\mu = 4$  it goes to equation (6a).

Thus the generalized Maxwell's equations are represented by only two equations involving operations on the components of the generalized form of the electromagnetic field tensor. The generalized form of the four-vector force density or Minkowski force can be given by

$$f_\mu = \frac{1}{c} F_{\mu\nu} j_\nu \quad (16)$$

With the help of equation (16) it converts to

$$f_\mu = \frac{1}{c} F_{\mu\nu} \frac{\partial F_{\nu\sigma}}{\partial x_\sigma} \quad (17)$$

Calculating its components we find

$$\mathbf{f} = \left( \mathbf{E} + \frac{\mathbf{u} \times \mathbf{H}}{c} - \frac{\mathbf{u}}{c} E_0 \right) \quad (18)$$

and

$$f_4 = \frac{i}{c} (\mathbf{E} \cdot \mathbf{j} - c\rho E_0) \quad (19)$$

Equation (18), which is the space part, is the generalized form of the Lorentz force and it seems that the scalar field decelerates the charge and the declaration produced by it is proportional to the velocity of the charge. Thus the electric vector field accelerates the charge while the electric scalar field decelerates. In absence of the scalar field the generalized form of the force acquires its previous form given by Lorentz and hence it the proper generalization. The space part (equation 18) represents the rate of change of mechanical momentum per unit volume and the time part (equation 19) represents the rate of change of mechanical energy per unit volume, i.e., the rate at which the field does work on the charges per unit volume.

In a general system consisting electric charged particles moving in a four component electromagnetic field, a portion of the energy of the system may be identified as kinetic energy of the particles, and the remainder may be viewed as potential energy or field energy. In the previous section we found that the relation connecting energy density and energy flow is

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \mathbf{S} = c\rho E_0 - \mathbf{j} \cdot \mathbf{E} \quad (20)$$

where

$$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H} + \mathbf{E}E_0) \quad (21)$$

is the generalized form of the Poynting vector and

$$\epsilon = \frac{1}{8\pi} [E^2 + H^2 + E_0^2] \quad (22)$$

is the generalized form of the field energy density. These results can also follow in an elegant way from consideration of the generalized form of the electromagnetic energy-momentum tensor defined by

$$T_{\mu\nu} = \frac{1}{4\pi} \left[ F_{\mu\sigma} F_{\sigma\nu} - F_{\mu\mu} F_{\nu\nu} + \frac{1}{4} \delta_{\mu\nu} F_{\lambda\rho} F_{\lambda\rho} - \frac{1}{2} \delta_{\mu\nu} F_{\mu\nu} F_{\mu\nu} \right] \quad (23)$$

By calculating individual components we obtain

$$T_{ij} = \frac{1}{4\pi} \left[ E_i E_j + H_i H_j - E_0 H_k - \frac{1}{2} \delta_{ij} (E^2 + H^2 + E_0) \right] \quad (24)$$

$$T_{4k} = -\frac{1}{4\pi} [\mathbf{E} \times \mathbf{H} + \mathbf{E} E_0] \quad (25a)$$

$$T_{k4} = -\frac{1}{4\pi} [\mathbf{E} \times \mathbf{H} - \mathbf{E} E_0] \quad (25b)$$

and

$$T_{44} = \frac{1}{8\pi} [E^2 + H^2 + E_0^2] \quad (25c)$$

This electromagnetic energy-momentum tensor is not symmetric tensor as in case without generalization. This is due to presence of the scalar field. We can consider the Poynting vector  $\mathbf{S}_+$  as sum of contributions from the vector fields and the scalar field and another Poynting vector  $\mathbf{S}_-$  as the difference of these contributions then we write the above tensor as

$$\{T\} = \begin{Bmatrix} \{T^m\} & \left(-\frac{i}{c} \mathbf{S}_+\right) \\ \left(-\frac{i}{c} \mathbf{S}_-\right) & \in \end{Bmatrix} \quad (26)$$

where  $\{T^m\}$  is the generalized form of the Maxwell's stress tensor. The two forms of the Poynting vectors are

$$\mathbf{S}_+ = -\frac{i}{4\pi} [\mathbf{E} \times \mathbf{H} + \mathbf{E} E_0] \quad (27a)$$

$$\mathbf{S}_- = -\frac{i}{4\pi} [\mathbf{E} \times \mathbf{H} - \mathbf{E} E_0] \quad (27b)$$

The Lorenz force density in terms of the energy-momentum tensor has the following form.

$$\frac{\partial T_{\nu\sigma}}{\partial x_\sigma} = f_\mu^m \quad (28)$$

The fourth component of which gives equation (20) and is the energy conservation law in a four component electromagnetic field. Thus the generalized Maxwell's theory can be formulated in tensor form in the Minkowski space in similar way as in the case of without generalization.

### 3. GDME in Tensor Form - Magnetic Charges as Source of Fields

GDME for magnetic monopoles as source of fields are

$$\nabla \cdot \mathbf{E} = 0 \quad (29a)$$

$$\nabla \cdot \mathbf{H} = \frac{1}{c} \frac{\partial H_0}{\partial t} + 4\pi \rho^m \quad (29b)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} + \nabla H_0 = -\frac{4\pi}{c} \mathbf{j}^m \quad (29c)$$

$$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = 0 \quad (29d)$$

The solutions are

$$\mathbf{E} = -\nabla \times \mathbf{A}^m \quad (30a)$$

$$\mathbf{H} = -\nabla \phi^m - \frac{1}{c} \frac{\partial \mathbf{A}^m}{\partial t} \quad (30b)$$

$$H_0 = \nabla \cdot \mathbf{A}^m + \frac{1}{c} \frac{\partial \phi^m}{\partial t} \quad (30c)$$

The four current densities and the four potentials have the forms as

$$j_\mu^m = (\mathbf{j}^m, i c \rho^m), \quad A_\mu^m = (\mathbf{A}^m, i \phi^m) \quad (31)$$

The generalized form of the electromagnetic field tensor is then

$$F_\mu^m = \frac{\partial A_\nu^m}{\partial x_\mu} - \frac{\partial A_\mu^m}{\partial x_\nu} - \delta_{\mu\nu} \delta_{\alpha\beta} \frac{\partial A_\beta^m}{\partial x_\alpha} \quad (32)$$

In this equation the last term is additional one used for the scalar field  $H_0$ . The superscript m used here is to represent the field tensor for fields created by magnetic monopoles 2<sup>nd</sup> not as a tensor index.

Calculating all the components of the field tensor, we obtain

$$F_{\mu\nu}^m = \begin{Bmatrix} -H_0 & -E_z & E_y & -iH_x \\ -E_z & -H_0 & -E_x & -iH_y \\ -E_y & -E_x & -H_0 & -iH_z \\ iH_x & iH_y & iH_z & H_0 \end{Bmatrix} \quad (33)$$

The two homogeneous Maxwell's type Dirac's equations (29a) and (29b) can be represented by the equation.

$$\frac{\partial F_{\lambda\nu}^m}{\partial x_\nu} + \frac{\partial F_{\mu\nu}^m}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}^m}{\partial x_\mu} = 0 \quad (34)$$

The other two inhomogeneous equations can be obtained from

$$\frac{\partial F_{\mu\nu}^m}{\partial x_\nu} = \frac{4\pi}{c} j_\mu^m \quad (35)$$

The generalized form of the four-vector force density or Minkowski force on a magnetic monopole is then

$$f_\mu^m = \frac{1}{c} F_{\mu\nu}^m j_\nu^m \quad (36)$$

With the help of equation (35) it converts to

$$f_\mu^m = \frac{1}{c} F_{\mu\nu}^m \frac{\partial F_{\nu\sigma}^m}{\partial x_\sigma} \quad (37)$$

In component form we get

$$f^m = \rho^m \left( \mathbf{H} - \frac{\mathbf{u} \times \mathbf{E}}{c} - \frac{\mathbf{u}}{c} H_0 \right) \quad (38)$$

$$\text{and } f_4^m = \frac{i}{c} (\mathbf{H} \cdot \mathbf{j}^m - c \rho^m H_0) \quad (39)$$

Form of the Lorentz force (equation 38) it seems that the scalar field  $H_0$  decelerates the magnetic monopole similar to that of electric scalar field decelerates the electric charge and the deceleration produced by it is proportional to the velocity of the monopole. Thus the magnetic vector field accelerates the magnetic monopole while the magnetic scalar field decelerates. In absence of the scalar field the generalized form of the force acquires its original form.

The corresponding electromagnetic energy-momentum tensor can be defined by

$$T_{\mu\nu}^m = \frac{1}{4\pi} \left[ F_{\mu\sigma}^m F_{\sigma\nu}^m - F_{\mu\mu}^m F_{\nu\nu}^m + \frac{1}{4} \delta_{\mu\nu} F_{\lambda\rho}^m F_{\lambda\rho}^m - \frac{1}{2} \delta_{\mu\nu} F_{\mu\nu}^m F_{\mu\nu}^m \right] \quad (40)$$

By calculating individual components we obtain

$$T_{ij}^m = \frac{1}{4\pi} \left[ E_i E_j + H_i H_j - H_0 E_k - \frac{1}{2} \delta_{ij} (E^2 + H^2 + H_0^2) \right] \quad (41)$$

$$T_{4k}^m = -\frac{i}{4\pi} [\mathbf{E} \times \mathbf{H} + \mathbf{H}H_0] = -\frac{i}{c} \mathbf{S}_+ \quad (42a)$$

$$T_{k4}^m = -\frac{i}{4\pi} [\mathbf{E} \times \mathbf{H} - \mathbf{H}H_0] = -\frac{i}{c} \mathbf{S}_- \quad (42b)$$

and

$$T_{44}^m = -\frac{i}{8\pi} [E^2 + H^2 + \mathbf{H}H_0] = \epsilon \quad (42c)$$

This electromagnetic energy-momentum tensor is also not symmetric. The Lorentz force density in terms of the energy-momentum tensor has the following form.

$$\frac{\partial T_{\nu\sigma}^m}{\partial x_\sigma} = f_\nu^m \quad (43)$$

The fourth component of which gives

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \mathbf{S} = c\rho^m H_0 - \mathbf{j}^m \cdot \mathbf{H} \quad (44)$$

which is the energy conservation law in a four component electromagnetic field when the field does work on magnetic monopoles.

#### 4. GDME in tensor form - electric as well as magnetic charges as sources of fields

If we consider the combined form of the generalized equations i.e. the generalized Dirac-Maxwell equations (equation 4) and their solutions (equations 2 and 5) then they can also be expressed in tensor form. For this we consider the four components of the electric and magnetic fields as

$$E_\mu = (\mathbf{E}, iE_0) \quad H_\mu = (\mathbf{H}, iH_0) \quad (45)$$

Similarly, the four currents and the four potentials are expressed as

$$j_\mu^{e,m} = \frac{4\pi}{c} (\mathbf{j}^{e,m}, ic\rho^{e,m}) \quad (46a)$$

$$A_\mu^{e,m} = (\mathbf{A}^{e,m}, i\varphi^{e,m}) \quad (46b)$$

$$\text{with } \partial_\mu = \left( \nabla, -\frac{i}{c} \frac{\partial}{\partial t} \right) \quad (47)$$

If we define a tensor by

$$i_{\mu\nu} = \partial_\mu H_\nu - \partial_\nu H_\mu + i\epsilon_{\mu\nu\alpha\beta} \partial_\alpha E_\beta \quad (48)$$

Then GDME given by equations (4) can be represented in tensor form by

$$i_k^e = j_{ij}^e, \quad i_k^m = -ij_{4k}^m \quad (49)$$

$$i_4^e = i\partial_\mu E_\mu, \quad i_4^m = i\partial_\mu H_\mu$$

where  $\epsilon_{\mu\nu\alpha\beta}$  is the fully antisymmetric symbol (Levi-Civita density) and  $\mu, \nu, \alpha, \beta = 1, 2, 3, 4$ . Similarly, further we define a tensor by

$$F_{\mu\nu} = \partial_\mu A_\nu^e - \partial_\nu A_\mu^e + i\epsilon_{\mu\nu\alpha\beta} \partial_\alpha A_\beta^m \quad (50)$$

Obviously, the tensor form for the solutions of the GDM equations can be give by

$$H_\alpha = F_{ij}, \quad H_k = -iF_{4k}$$

$$E_0 = \partial_\mu A_\mu^e, \quad H_0 = \partial_\mu A_\mu^m \quad (51)$$

The wave equations are given by

$$\square F_{\mu\nu} = -\left( \partial_\mu j_\nu^e - \partial_\nu j_\mu^e + i\epsilon_{\mu\nu\alpha\beta} \partial_\alpha j_\beta^m \right)$$

$$\square E_0 = \partial_\mu j_\mu^e, \quad \square H_0 = \partial_\mu j_\mu^m \quad (52)$$

These tensor equations are similar to those obtained by Cabibbo-ferrari tensor [25], which can also be compared with Mognani-Recami tensor [29-30], who considered tachyon charges as magnetic one.

#### 5. Discussion

Writing the field equations in tensor forms is to reduce the number of field equations. Tensor form of the usual Maxwell's field equations is familiar to every one in physics. However, generalization of Maxwell's equation in order to accommodate the counterpart of the electric charges as magnetic monopoles is a significant work which made Maxwell's equations symmetric. Further generalization of Dirac-Maxwell's equations in order to accommodate the change in charge on the particles, electric as well as magnetic, is also a substantial work. These generalized Dirac-Maxwell's equations and their solutions and fields generated by such particle are studied in [8-10]. A new phenomenon, called inversion in the field at critical distance from a stationary charge whose charge is decreasing with time is explored by the generalized theory. The present work successfully converts the field equations involved in such generalized theory into tensor forms.

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