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Study of Methods for Solving Second-Order ODEs with Constant Coefficient

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Abstract: ***The paper provides an in-depth analysis of various methods for solving second-order ordinary differential equations (ODEs) with constant coefficients, categorizing solutions based on the roots of the characteristic equation. Different real roots, complicated conjugate roots, and repetitive roots are the three primary types of roots that are covered by this. In the case of distinct real roots, the solution is a linear combination of exponential functions. On the other hand, complex conjugate roots result in solutions that include exponential and trigonometric functions. In order to assist students and researchers who are interested in understanding and solving second-order ordinary differential equations with constant coefficients with a roadmap that is both clear and succinct, the purpose of this work is to create such a guide. In the case of repeated roots, the solution uses a mix of exponential functions and a linear term to solve the problem. The formulation of the characteristic equation, the process of solving for roots, and the application of beginning conditions to find particular constants are all topics that are covered in this work. These approaches are shown using practical examples, which emphasize their usefulness in domains such as engineering, physics, and applied mathematics: these examples are offered to demonstrate these methods. In order to ensure that a full grasp of the topic is achieved, the study places an emphasis on the significance of these approaches in modeling a variety of physical processes and provides an organized approach to solving second-order ordinary differential equations.***

Keywords: Second-order differential equations, Linear differential equations, Delay Differential Equations (DDEs)

1. Introduction

A second order differential equation is a type of differential equation that involves a function and its second derivative, but excluding any higher-order derivatives of the function. The types of derivatives can vary based on the magnitude of the derivative's power and the functions being used. The auxiliary equation can be utilized to solve these differential equations. Now, we will examine several distinct types of second order differential equations:

Linear second-order differential equation

A linear second order differential equation can be expressed in the form y" + p(x)y' + q(x)y = f(x), where the power of the second derivative y" is one, thereby making the equation linear. Examples of differential equations include y" + 6x = 5 and y" + xy' + y = 0.

Second order differential equation with constant coefficients

A second order differential equation y" + p(x)y' + q(x)y = f(x) is considered a second order homogeneous differential equation when f(x) is equal to zero, resulting in the equation y" + p(x)y' + q(x)y = 0. Examples of such equations include y" + y' - 6y = 0, y" - 9y' + 20y = 0, and so on.

Non-homogeneous second order differential equation

A non-homogeneous second order differential equation is defined as a differential equation of the form y" + p(x)y' + q(x)y = f(x), where f(x) is a non-zero function. Examples of differential equations include y" + y' - 6y = x and y" - 9y' + 20y = sin x.

Second order differential equation with constant coefficients

A second order differential equation with constant coefficients is defined by the equation y" + p(x)y' + q(x)y = f(x), where p(x) and q(x) are constant functions. Examples of differential equations include y" + y' - 6y = x and y" - 9y' + 20y = sin x.

Second order differential equation with coefficients that vary with the independent variable.

A second order differential equation with variable coefficients, denoted as y" + p(x)y' + q(x)y = f(x), is characterized by the non-constant functions p(x) and q(x), which are dependent on the variable x. Examples of differential equations include y" + xy' - y sinx = x and y" - 9x2y' + 2exy = 0.

The resolution of nonlinear equations f(x) = 0 is a key issue in numerical analysis, and numerous techniques have been devised to address it. The available methods vary from the dependable albeit often sluggish bisection approach to swifter and more advanced techniques such as the Newton-Raphson method, which may exhibit less predictability. In order to determine all the solutions of a real function f(x) within a specific interval, it is typically required to confine these solutions within bracketing intervals (which are suitable for the bisection method) or to obtain initial approximations for the roots that are accurate enough. These approximations can then be further improved using methods such as Newton-Raphson or other higher-order techniques(Yakimiw, 1996). The latter method is frequently employed to calculate the zeros of particular functions, such as orthogonal polynomials and Bessel functions(Temme, 1979).

Nevertheless, it can be quite difficult to obtain precise starting estimates for all the zeros, particularly when the function f(x) relies on numerous parameters. It is challenging to guarantee convergence for all parameter values and zeros. This matter is emphasized within the framework of Legendre polynomials that possess only one parameter(Petras, 1999).

Novel approaches have been devised that necessitate reduced levels of precise data regarding the exact whereabouts of zeros. For example, specific specialized techniques can be employed for specific sorts of functions without requiring the use of brackets or pre-established approximations.

A significant category of functions consists of solutions to linear homogeneous second-order difference equations. There are specific techniques available for certain solutions. Significantly, in the case of orthogonal polynomials pk(x) where k represents the degree, the zeros can be calculated as eigenvalues of real tridiagonal symmetric matrices. The elements of these matrices can be obtained from the coefficients of the three-term recurrence relation: $p_{n+1}(x) = (A_n x + B_n)p_n(x) + C_n p_{n-1}(x)$.

For some nonpolynomial functions, the zeros can alternatively be estimated as eigenvalues of real tridiagonal symmetric matrices. This method is suitable for finding solutions to three-term recurrence relations, such as regular Bessel functions and Coulomb wave functions.

Another important category consists of functions that fulfill first-order linear homogeneous difference-differential equations (DDEs). The fixed point methods (FPMs) obtained by approximating the integration of Riccati equations linked to these DDEs are globally convergent and exhibit second-order accuracy. These methods can accurately calculate all real zeros inside a certain interval. They are applicable in scenarios where matrix eigenvalue methods are successful, and can be utilized in a wider range of situations. By integrating these Delay Differential Equation (DDE) approaches with Finite Point approaches (FPMs) for identifying extreme values (using related Ordinary Differential Equations, ODEs), it is possible to create second-order techniques for calculating the roots of DDE solutions and their initial derivatives. To obtain a comprehensive examination of DDE methods and matrix eigenvalue methods, of the provided reference.

The third collection of functions, which is likely the most important, consists of solutions to linear second-order homogeneous ordinary differential equations (ODEs). This collection encompasses all functions that are dealt with using matrix Eigen value and DDE (delay differential equation) methods, among others.

Here, we provide a fourth-order technique that achieves global convergence when certain monotonicity conditions are met. This exceptional characteristic is of great importance for a method of fourth order. The methodology relies on the estimation of a Riccati equation linked to the ordinary differential equation (ODE). It follows a similar principle to Newton's method, where the logarithmic derivative of the function in question is computed. The coefficients of the ordinary differential equation (ODE) are also employed. This iterative approach enables the development of an algorithm that can accurately calculate all the zeros within any given real interval. Preliminary approximations for each root are superfluous given the method's universal convergence, with the sole prerequisite being awareness of the monotonicity of the ODE coefficients.

The second-order ordinary differential equation in normal form is given by: $y''(x) + A(x)y(x) = 0$, Prior knowledge of the sign of A'(x) and its potential variations is necessary. This method offers a highly effective approach for calculating the actual roots of several significant functions. It usually attains a precision of 100 digits in just 4-5 iterations per root, surpassing other efficient global approaches that rely on DDEs. If good initial approximations for the zeros are available, they can be utilized to further improve the approach. Nevertheless, the newfound method's rapidity and straightforwardness render it a universal approach for solving any ordinary differential equation, regardless of the particular initial values.

The paper is organized in the following manner:
1) Section 2: A fourth-order fixed-point algorithm is devised, demonstrating monotonic convergence features where A(x) is monotonic. The method's global convergence is established under specific constraints on the rate of variation of A(x).
2) Section 3: The fixed-point method is adapted to ensure a globally convergent iteration. A precise algorithm is formulated to calculate all the zeros inside a certain interval, ensuring complete confidence. Additionally, a technique for computing the zeros of the derivative is also examined.
3) Section 4: This section focuses on analyzing the nonlocal behavior of the algorithm. Specifically, it provides a detailed examination of the scenario where the equation y''(x) + x^{-m}y(x) = 0 is considered, with m being a real number.
4) Section 5 showcases the algorithm's performance by providing examples of various mathematical functions such as classical orthogonal polynomials (Hermite, Laguerre, and Jacobi), Coulomb wave functions, Bessel functions with real or imaginary orders and variables, and conical functions.

This meticulous methodology guarantees the reliable and effective calculation of actual zeros for a wide range of significant functions.

2. Objectives

The aim of this article is to provide a clear and concise guide for students and researchers interested in understanding and solving second-order ordinary differential equations with constant coefficients.

3. Literature Review

(Santra, Bazighifan, Ahmad, & Yao, 2020) Second-order differential equations arise in various physical phenomena, including fluid dynamics, electromagnetism, acoustic vibrations, and quantum physics. This work establishes the necessary and sufficient conditions for the solutions to second-order half-linear delay differential equations of the form represented by the mathematical equation, assuming that the integral of the reciprocal of the function ς(η) with respect to η, from negative infinity to positive infinity, is equal to infinity. We examine two scenarios: one where a < cj and another where a > cj. Here, a and cj represent the result of dividing two positive odd integers. The usefulness and application of the outcome are demonstrated through two cases.

(Santra, Bazighifan, Ahmad, & Chu, 2020) Second-order differential equations are prevalent in diverse fields such as physics, mathematics, and engineering. This study establishes the necessary and sufficient conditions for oscillations of solutions to second-order half-linear delay differential equations of the type ςyu′ya′+pyucϑy=0, for y≥y0, under the assumption that the integral of ∞ςη-1/a diverges. Two situations are examined: one where ac. Here, a and c represent the ratios of two positive odd integers. The usefulness and application of the outcome are demonstrated through the provision of two examples.

(Santra et al., 2021) Neutral differential equations are commonly encountered in continuous applications in domains like as electrodynamics, neural networks, quantum mechanics, electromagnetism, temporal symmetry, and fluid dynamics. These equations are essential for accurately describing a wide range of problems and phenomena. Due of its importance, the study of the qualitative behavior of solutions to these equations is highly interesting. This paper establishes novel criteria for the oscillatory nature of solutions to second-order delay differential equations with sub-linear neutral components. These findings bolster and expand upon the current outcomes in the literature. In order to validate our primary findings, we offer a concrete illustration and introduce an unresolved issue that warrants additional exploration. This emphasizes the practical significance and the possibilities for future scholarly inquiry in this field.

(Džurina & Jadlovská, 2020) This study introduces novel requirements for the oscillation of all solutions to a second-order half-linear delay differential equation in noncanonical form. These criteria replace the previously assumed condition that the delay function is nondecreasing. We confirm that the oscillation constant is optimal, indicating that the theorem's validity cannot be maintained if the strict inequality is substituted with a nonstrict one. This conclusion is innovative and surpasses all previously reported results in the literature, even in the linear situation. Our approach has a significant benefit in that the proof is straightforward and relies only on the progressively enhanced monotonicities of a positive solution. This makes our conclusions reliable and easy to understand.

(Verma et al., 2020) Nonlinear singular differential equations are used to simulate various real-life problems. This paper focuses on analyzing a specific category of nonlinear singular differential equations. We thoroughly examine several characteristics of these equations and conduct an extensive review of existing research on the topic. Solving nonlinear singular differential equations is a challenging task, because in most circumstances, an exact solution does not exist. Given the absence of an exact solution, it is logical to investigate the presence of both an analytical solution and a numerical solution. This survey examines the two aspects of nonlinear singular boundary value problems (SBVPs) and explores various analytical and numerical techniques that have been developed to address a specific class of nonlinear singular differential equations. These equations are of the form (p(x)y'(x))' = q(x) f (x, y, py') for x ∈ (0, b), subject to appropriate initial and boundary conditions. The monotone iterative technique has attracted significant attention in the past two decades and has been integrated with many other current techniques. Additionally, a compilation of Subject-Verb-Object-Prepositional Phrase structures is included, which will greatly assist researchers engaged in this field.

4. Methodology

The article explores the characteristic equation and its roots, which play a vital role in determining the nature of the solutions to second-order differential equations. The methodology is categorized into three primary scenarios, determined by the characteristics of the roots of the characteristic equation:

1) Distinct and Real Roots:

- When the characteristic equation produces two unique and real roots, the solution to the differential equation can be expressed as a linear combination of two exponential functions.
- The equation can be written in the generic form

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t})$$

Where, C_1 and C_2 are constants and r_1 and r_2 are the exponential growth rates.{r_2t}The equation

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t})$$

represents a solution with separate roots r1 and r2. The constants C_1 and C_2 are determined by the beginning circumstances.

- An example is given to demonstrate the process of finding the roots and formulating the general solution.

2) Complex Conjugate Roots:

- When the characteristic equation produces complex conjugate roots, the solution involves exponential and trigonometric functions.
- The equation can be expressed as

$$y(t) = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

Where, α and β are constants. The function y(t) is equal to the exponential of alpha times t, multiplied by the sum of C_1 times the cosine of beta times t and C_2 times the sine of beta times t.The function y(t) can be expressed as eαt(C_1cos(βt)+C_2sin(βt)), where α±iβ are the complex roots.

- An illustrative example is presented to showcase the process of obtaining the overall solution from intricate roots and implementing basic conditions.

3) Repeated Roots:

- When the characteristic equation produces a root that occurs more than once, the solution involves an exponential function and a linear term.
- The equation is given by

$$y(t) = (C_1 + C_2 t)e^{rt}$$

Where, r is the repeating root.

- An example is given to illustrate the process of identifying the recurring root and proposing the overall solution.

The general solution for each situation is derived by following a set of precise stages. These steps include formulating the characteristic equation, solving for the roots, and applying beginning conditions to obtain the individual constants in the solution. Illustrative examples are provided to demonstrate the practical implementation of these strategies, so ensuring a thorough comprehension of the process for finding a solution.

5. Result

Homogeneous Linear ODE Coefficients with Constant

This section focuses on the specific scenario of a second-order homogeneous linear differential equation when all the coefficients are real constants. Specifically, our focus will be on the equation.

This section specifically addresses the second-order homogeneous linear differential equation (LDE), in which all coefficients are constant and have real values. Specifically, we will exclusively examine equations that adhere to the following format:

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

Hence, we will seek solutions to equation (1) in the format of $y=e^{mx}$, where we will select the constant m such that $y=e^{mx}$ fulfills the equation. If we assume that mx is a valid solution for a specific value of m, we can express it as a y

$$\frac{dy}{dx} = me^{mx}, \frac{d^2y}{dx^2} = m^2e^{mx} \quad (1)$$

$$a_0(x)m^2e^{mx} + a_1(x)me^{mx} + a_2(x)e^{mx} = 0$$

Or, (m+4) (m+2)=0

Given that $e^{mx} \neq 0$, we can derive a polynomial equation in the variable m;

$$a_0(x)m^2 + a_1(x)m + a_2(x) = 0 \quad (2)$$

The equation presented above is commonly known as the auxiliary equation or the characteristic equation of the provided differential equation (1). When calculating the auxiliary equation, three possible circumstances may occur.

a) All the roots are unique and exist in the real number system.
b) All of the roots are real, however there are some that repeat.
c) All of the roots are complex numbers with no real part.

Distinct Real Roots

if m_1, m_2 are different roots then

$y = e^{m_1x}, y = e^{m_2x}$

Are independent solutions of (1). Therefore, the general solution of (1) is.

$y = c_1e^{m_1x} + c_2e^{m_2x}$

where c_1, c_2 are arbitrary constants.

Ex.

$$2\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 16y = 0$$

The auxiliary equation is

$$2m^2 - 12m + 16y = 0$$

Hence,

(m-4) (2m-4) = 0

M=4,2

The roots of the equation are both real and distinct. The solutions to the equation are e^{4x} and e^{2x}. The general solution can be expressed as:

$y = c_1e^{4x} + c_2e^{2x}$ where C_1 and C_2 are arbitrary constants.

Non-Homogeneous Linear ODE with Constant Coefficients

Where the coefficients a_0, a_1, a_2 are constants but the non homogeneous term F in general a non constant function of x.The general solution of the above equation may be written, $y = y_c + y_p$ where y_c is the general solution of the corresponding homogeneous equation (1) with F replaced by zero and y_p is called the complementary function, and it is a solution that contains no arbitrary constant. On the other hand, any solution of equation (1) that does not contain arbitrary constants is known as a particular integral.

4.1 Case-1: if F(x) = x, polynomial in x then

$$y_p = \frac{1}{f(D)}x = [f(D)]^{-1}X$$

This can be applying binomial expansion $[f(D)]^{-1}$ and multiplying term by term. Sometimes the expansions are made by using partial fraction.

Ex.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + 5$$

The auxiliary equation is

$m^2 +3m+2=0$

or(m+2)(m+1)=0

m=-2,-1

The roots are distinct and real. Thus e^{-x} and e^{-2x} are the solutions that satisfy the equation, and the solution that is not associated with any particular initial condition, can be expressed as the complementary solution.

$$y_c = c_1e^{-x} + c_2e^{-2x}$$

Where C_1 and C_2 are arbitrary constants.

The particular solution is,

$$y_p = Ax + B$$

Where A, B are constant undetermined coefficient to be determined. Taking the derivative of the equation yields:

$y_p = 0$

$y_p = A$

Substituting these in equation we obtain,

$0 + 3(A) + 2(Ax + B) = 4x + 5$

$Or \;\; 3A + 2B + 2Ax = 4x + 5$

Equating the coefficient of x and constant term we obtain,

$3A + 2B = 5$

$2A = 4$

After solving

$A = 2$ and $B = \frac{-1}{2}$

Substituting obtain

$y = y_c + y_p$

$$y = c_1 e^{-x} + c_2 e^{-2x} + 2x - \frac{1}{2}, where\ C_1 and\ C_2 are\ arbitrary\ constants.$$

An initial value problem

The initial problem involves determining the integration constants of the general solution of a differential equation. This is done by substituting the variables and their derivatives in the solution with the given initial values, and then solving the resulting equation to find the required constant.

Ex.

$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0\ Where\ y(0) = 1\ and\ y(0) = 6$

The auxiliary equation is

$m^2 - 7m + 12 = 0$

Or (m-3) (m-4)=0

M=3,4

The roots are real and distinct. Thus e^{4x} and e^{3x} the solutions we have obtained are valid, and it is possible to express the complementary solution in the following manner:

$y = c_1 e^{4x} + c_2 e^{3x}$

Where C_1 and C_2 are arbitrary constant from this we find

$\frac{dy}{dx} = 4c_1 e^{4x} + 3c_2 e^{3x}$

We implement the initial conditions. Implementing condition y (0)=1 to equation (3.4) and $y(0) = 6$ to equation (3.5) we find

$c_1 + c_2 = 1; \quad 4c_1 + 3c_2 = 6$

Solve this we find, $c_1 = 6\ and\ c_2 = -5$

Replacing c1 and c2 in equation

$y = 6e^{4x} - 5e^{3x}$

6. Discussion

The analysis of techniques for solving second-order ordinary differential equations with constant coefficients emphasizes the strength and adaptability of various strategies based on the characteristics of the roots. Equations that result in separate and actual solutions are represented as combinations of exponential functions, which can be described in clear and analytical forms. When the roots are complex conjugates, the solutions involve both exponential and trigonometric functions, which indicate oscillatory activity. When there are multiple identical answers, the solutions involve a combination of exponential functions and polynomial terms to ensure that all possible circumstances are accounted for completely. The approaches given guarantee precise root determination by utilizing characteristic equations to convert intricate differential equations into accessible algebraic forms. The effectiveness of these strategies is demonstrated through practical examples and detailed derivations, highlighting their relevance in diverse domains such as physics, engineering, and applied mathematics. This complete method not only promotes comprehension but also improves the capacity to address real-world situations represented by second-order ordinary differential equations.

7. Conclusion

Research on techniques for solving second-order ordinary differential equations (ODEs) with constant coefficients offers a complete manual for academics and students. It groups the answers according to the kind of the roots of the characteristic equation: repeating roots, complicated conjugate roots, and separate, real roots. Solues for different real roots are stated as a linear combination of exponential functions. The solutions where the roots are complex conjugates combine trigonometric and exponential functions. Repeated roots produce answers combining linear terms and exponential functions. Deriving characteristic equations, solving for roots, and using beginning conditions to find particular constants comprise the method under discussion in this work. This method guarantees a clear knowledge of the method needed to solve second-order ODEs by means of structure. The relevance of second-order differential equations in modeling different physical events and their applications in fields including fluid dynamics, electromagnetism, and quantum physics is underlined by illustrative examples throughout the study of these approaches.

References

[1] Džurina, J., & Jadlovská, I. (2020). A sharp oscillation result for second-order half-linear noncanonical delay differential equations. *Electronic Journal of Qualitative Theory of Differential Equations*. https://doi.org/10.14232/ejqtde.2020.1.46

[2] Petras, K. (1999). On the computation of the Gauss-Legendre quadrature formula with a given precision. *Journal of Computational and Applied Mathematics*. https://doi.org/10.1016/S0377-0427(99)00225-3

[3] Santra, S. S., Bazighifan, O., Ahmad, H., & Chu, Y. M. (2020). Second-Order Differential Equation: Oscillation Theorems and Applications. *Mathematical Problems in Engineering*. https://doi.org/10.1155/2020/8820066

[4] Santra, S. S., Bazighifan, O., Ahmad, H., & Yao, S. W.

(2020). Second-Order Differential Equation with Multiple Delays: Oscillation Theorems and Applications. *Complexity*. https://doi.org/10.1155/2020/8853745

[5] Santra, S. S., Bazighifan, O., & Postolache, M. (2021). New conditions for the oscillation of second-order differential equations with sublinear neutral terms. *Mathematics*. https://doi.org/10.3390/math9111159

[6] Temme, N. M. (1979). An algorithm with ALGOL 60 program for the computation of the zeros of ordinary bessel functions and those of their derivatives. *Journal of Computational Physics*. https://doi.org/10.1016/0021-9991(79)90134-7

[7] Verma, A. K., Pandit, B., Verma, L., & Agarwal, R. P. (2020). A review on a class of second order nonlinear singular BVPs. In *Mathematics*. https://doi.org/10.3390/MATH8071045

[8] Yakimiw, E. (1996). Accurate computation of weights in classical Gauss-Christoffel quadrature rules. *Journal of Computational Physics*. https://doi.org/10.1006/jcph.1996.0258