International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2020): 7.803

Total k-rainbow Domatic Number

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Abstract: ***The Total k-rainbow domination number is defined by considering k types of guards or set of k colors. A location which does not have any type of guard (colour) assigned, should have all type of guards (colours) in its immediate surrounding location to protect it. For a positive integer k, a function $f: V(G) \to P(\{1,2,\dots,k\})$ is said to be a total k-rainbow dominating function if $\forall\, v \in V(G), \bigcup_{u\in N[v]} f(u) = \{1,2,\dots,k\}$ where $f(u)$ is nonempty subset of $\{1,2,\dots,k\}$ and $N[v]$ is a closed neighborhood of v. A set $\{f_1, f_2, \dots, f_d\}$ of total k-rainbow dominating function of a graph G with the property that $\sum_{i=1}^{d} |f_i(v)| \le k$ for each $v \in V(G)$, is called a total k-rainbow dominating family (functions) on G. The maximum number of functions in a total k-rainbow dominating family (TkRD family) on G is called the total k-rainbow domatic number of G, is denoted by $d_{trk}(G)$. In this paper we initiate the study of total k-rainbow domatic number in graphs and we obtain $d_{trk}(K_n) = min\{n,k\}$, $d_{trk}(C_n) \le 3$. We also proved some bounds for $d_{trk}(G)$.***

Keyword: k-rainbow domination number, Total k-rainbow domination number, k-rainbow domatic number, Total k-rainbow domatic number

1 Introduction

Let G be a simple graph with vertex set $V = V(G)$ and the edge set $E = E(G)$. The number of vertices in the graph G is known as order of G and is denoted by $n = n(G)$. $\forall v \in V(G)$, the open neighborhood $N(V)$ is the set $\{u \in V(G); uv \in E(G)\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup v$. The degree of a vertex $v(G)$, $d(v)$ is the number of edges incident on the vertex v. The minimum and maximum degree of a graph G is denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ respectively. Tree T is a connected acyclic graph. We write K_n for complete graph of order n, C_n for a cycle of length n, P_n for path of length n and W_n for wheel graph of order n. We follow [1] and [2] for notation and graph theory terminology.

M. A. Henning [2] introduced the concept of k-rainbow domination number by considering mathematical model of assigning guards to each location from k different type of guards. According to him a location which is not having any type of guards need to have all type of guards in its neighboring location.

Definition 1.1. Let G be a graph and f be a function that assigns to each vertex a set of guards chosen from the set $\{1,2,\dots,k\}$ i.e. $f : V(G) \to P(\{1,2,\dots,k\})$. If for each vertex $v \in V(G)$ such that $f(V) = \emptyset$ we have

$\bigcup_{u\in N(v)} f(u) = \{1,2,\dots,k\}$,

then f is called the k-rainbow dominating function(kRDF) [4]. The weight $w(f)$ of k-rainbow domination number is defined by, $w(f) = \sum_{v\in V} |f(v)|$. The minimum weight a kRDF is called k-rainbow domination number and is denoted by $\gamma_{rk}(G)$.

The study of total k-rainbow domination number is introduced by P. P. Kumbargoudra and J. V. Kureethara. According to total k-rainbow dominating function each and every location is secured by all type of guards. A location which is not having any type of guards should have that type of guard in its immediate neighbouring location.

Definition 1.2. For a positive integer k, a total k-rainbow dominating function (TkRDF) of a graph G is defined in [?] as a function $f : V(G) \to P(\{1,2,\dots,k\})$ that assigns to each vertex a nonempty subset of a set $S = \{1,2,\dots,k\}$ i.e.,

$$\forall v \in V(G), \bigcup_{u\in N[v]} f(u) = S.$$

The weight $w(f)$ of total k-rainbow domination number is defined by,

$$w(f) = \sum_{v\in V} |f(v)|.$$

The minimum weight of a TkRDF is known as total k-rainbow domination number and is denoted by $\gamma_{trk}(G)$.

In a study S. M. Sheikholeslami and L. Volkmann [9] defined k-rainbow domatic number $d_{rk}(G)$. They found 2-rainbow domatic number for P_n, C_n and obtained bounds for 2-rainbow domatic number.

In this paper we introduce total k-rainbow damatic number and initiate the study of the total k-rainbow domatic number of some classes of graphs. We obtain basic bounds for the total k-rainbow domatic number of a graph.

Definition 1.3. A set $\{f_1, f_2, \dots, f_d\}$ of total k-rainbow dominating functions of a graph G with the property that $\sum_{i=1}^{d} |f_i(v)| \le k$ for each $v \in V(G)$, is called a total k-rainbow dominating family (functions) on G. The maximum number of functions in a total k-rainbow dominating family (TkRD family) on G is called the total k-rainbow domatic number of G, is denoted by $d_{trk}(G)$.

The total k-rainbow domatic number is well defined and $d_{trk}(G) \geq 1$, for all graph G, (4.1) since the set consisting of any total k-rainbow dominating function (TkRDF) forms a TkRD family on G.

2 Properties of Total k-rainbow Domatic Number

Proposition 2.1. If K_n is a complete graph of order $n \geq 3$, then $d_{trk}(K_n) = min\{n, k\}$.

Proposition 2.2. If P_n is a path of length n then $d_{trk}(P_n) = 2$.

Proposition 2.3. If W_n is a Wheel graph of order n then, $d_{trk}(W_n) = 3$.

Proposition 2.4. If T is a tree then, $d_{trk}(T) = 2$.

Theorem 2.5. If G is graph, then $d_{trk}(G) = 1$ if and only if G is empty.

Proof. If G is empty, then the mapping $f : V(G) \rightarrow P(\{1,2,\dots,k\})$ defined by $f(v) = \{1,2,\dots,k\}$ for each $v \in V$ is a unique k-rainbow dominating function on G and so $d_{trk}(G) = 1$.

Conversely, let $E(G) \neq \emptyset$ and let $uv \in E(G)$. Then the mappings $f : V(G) \rightarrow P(\{1,2,\dots,k\})$ defined by

$f(u) = \{1,2,\dots,k-1\}$ and $f(v) = k$ and $f(x) = \{1,2,\dots,k\}$ for each $x \in V - \{u,v\}$

And $f(u) = k$ and $f(v) = \{1,2,\dots,k-1\}$ and $f(x) = \{1,2,\dots,k\}$ for each $x \in V - \{u,v\}$ are k-rainbow dominating functions on G and $\{f,g\}$ is a k-rainbow dominating family on G. It follows that $d_{trk}(G) \geq 2$, and hence the proof.

Theorem 2.6. For any graph G, $2 \leq d_{trk} \leq k$.

Proof. Let v be a vertex of a graph G. Define the total k-rainbow dominating function $f : V(G) \rightarrow P(\{1,2,\dots,k\})$ by $f(v) = A$, where A is any nonempty subset of $\{1,2,\dots,k\}$.

Let f_1 be a total k-rainbow dominating function of G with for $\forall v \in V(G)$ such that $f_1(v) \neq \{1,2,\dots,k\}$. Now define another function $f_2(G)$ such that $\forall v \in V(G)$ such that $f_2(v) = f_1^c(v)$. Clearly f_2 is also total k-rainbow

dominating function. Hence $2 \leq d_{trk}$

If a set $\{f_1, f_2, \dots, f_d\}$ be a total k-rainbow dominating family on G. Since $\sum_{i=1}^{d}|f_i(v)| \leq k$ for each vertex $v \in V(G)$ and $|f_i(v)| \leq 1$. Hence $d_{trk} \leq k$.

Theorem 2.7. Let G be a graph with minimum degree $\delta = 1$ then,

$d_{trk}(G) = 2$.

Proof. Given G be a graph with $\delta = 1$. Let $v \in V(G)$ and $d(v) = 1$ and $N(v) = \{u\}$. Let $f_1 : V(G) \rightarrow \{1,2,\dots,k\}$ be a total k-rainbow dominating function with $f_1(v) = A$ where A is any nonempty subset of $\{1,2,\dots,k\}$ and $f_1(u) = S - A$.

Similarly define $f_2 : V(G) \rightarrow \{1,2,\dots,k\}$ a total k-rainbow dominating function with $f_2(v) = S - A$ and $f_2(u) = S$.
Clearly $|f_1(v)| + |f_2(v)| = k$. Hence $d_{trk}(G) = 2$.

Remark: Converse of the above theorem need not be true for example cycle C_n, where n is not multiple of 3. We also conclude proposition 2.2 and proposition 2.4.

Theorem 2.8. If G is graph of order n, then $\gamma_{trk}(G) \cdot d_{trk}(G) \leq kn$. If $\gamma_{trk}(G) \cdot d_{trk}(G) = kn$, then for each TkRD family $\{f_1, f_2, \dots, f_d\}$ on G with $d = d_{trk}(G)$, each function f_i is a $\gamma_{trk}(G)$–function, and $\sum_{i=1}^{d}|f_i(v)| = k$ for all $v \in V$.
Proof. Let $\{f_1, f_2, \dots, f_d\}$ be a TkRD family on G such that $d = d_{trk}(G)$.

Then,

$$d.\gamma_{trk}(G) = \sum_{i=1}^{d}\gamma_{trk}(G)$$
$$\leq \sum_{i=1}^{d}\sum_{v\in V}|f_i(v)|$$
$$= \sum_{v\in V}\sum_{i=1}^{d}|f_i(v)|$$

Since each edge can share a set of elements from $\{1,2,\dots,k\}$ to its two adjacent vertices so,

$$d.\gamma_{trk}(G) \leq \sum_{v\in V}\frac{k}{2} = \frac{k}{2}n \leq kn.$$

If $\gamma_{trk}(G) \cdot d_{trk}(G) = kn$, then two inequalities occurring in the proof become equalities. Hence for te kRD family $\{f_1, f_2, \dots, f_d\}$ on G and for each i,

$\sum_{v\in V}|f_i(v)| = \gamma_{trk}(G)$. Thus, each function f_i is a $\gamma_{trk}(G)$–function, and $\sum_{i=1}^{d}|f_i(v)| = \frac{k}{2}$ for all $v \in V$.

Theorem 2.9.

$$\text{For } n \geq 3, d_{trk}(C_n) = \begin{cases} 3 \text{ if } n \equiv 0 \pmod 3 \\ 2 \text{ Otherwise} \end{cases}$$

Proof. Let us consider S_1, S_2 *and* S_3 any three nonempty arbitrary subsets of the set $S = \{1, 2, \dots, k\}$ with,

$S_1 \cup S_2 \cup S_3 = S$ and $S_1 \cap S_2 \cap S_3 = \emptyset$.

Paper ID: SR22118125226 DOI: 10.21275/SR22118125226

The total k-rainbow dominating function[?] is defined as,

$$f(C_n) = \begin{cases} S_1, S_2, S_3, \ldots, S_1, S_2, S_3 \text{ if } n \equiv 0(mod\ 3) \\ S_1, S_2, S_3, \ldots, S_1, S_2, S_3, S_1 \cup S_2 \cup S_3 \text{ if } n \equiv 1(mod\ 3) \\ S_1, S_2, S_3, \ldots, S_1, S_2, S_3, S_1, S_2 \cup S_3 \text{ if } n \equiv 2(mod\ 3) \end{cases}$$

Case 1: $n \equiv 0\ (mod\ 3)$

Let v_i be any arbitrary vertex of a graph C_n then v_{i-1} and v_{i+1} will be its adjacent vertices. To define the total k-rainbow dominating function f_1, let us assign the subset S_1 to the vertex v_i and assign either of the subsets S_2 *or* S_3 to the adjacent vertices v_{i-1} *or* v_{i+1}.

By assigning different subsets of complete set S to the each of the vertices v_{i-1}, v_i, v_{i+1} in either clockwise or anti-clockwise direction we get the subset S_1 being replaced on the vertex v_i by either of the remaining subsets S_2 *or* S_3, the other two vertices will have the remaining subsets. This will lead us to three total k-rainbow dominating functions f_1, f_2 *and* f_3 for each subset S_1, S_2 *and* S_3 respectively.

Since n is a multiple of 3, we can conclude that each vertex $\{v_1, v_2, \ldots, v_n\}$ will have a complete set S by taking the union of functional value of f_1, f_2 and f_3. No additional total k-rainbow dominating function can be added to the total k-rainbow dominating family with the property $\sum_{i=1}^{d} |f_i(v)| \leq k$ apart from f_1, f_2 and f_3 for a given set S.

Case 2: $n \not\equiv 0\ (mod\ 3)$

From the equation (2) clearly we can define exactly two total k-rainbow dominating function f_1, f_2 in both the cases. Hence there exist exactly two functions with the property $\sum_{i=1}^{d} |f_i(v)| \leq k$.

Theorem 2.10. For any generalised Peterson graph $P(n, 3)$,

$$d_{trk}(P(n,3)) \leq 4.$$

Proof. Let $P(n, 3)$ be a Peterson graph with vertex set

$\{v_1, v_2, v_3, \ldots\ldots, v_n, u_1, u_2, u_3, \ldots\ldots, u_n\}$. Clearly every vertex in generalised Peterson graph is of degree three so the closed neighbourhood of an arbitrary vertex v_i *in* $V(P(n,3))$ is $\{v_{i-1}, v_i, v_{i+1}, u_i\}$. Let S_1, S_2, S_3, S_4 are the subset of the set $S = \{1, 2, 3, \ldots\ldots k\}$

The total k-rainbow dominating function is defined as [**?**], by rotating this assignment in clockwise direction we get,

$$d_{trk}(P(n,3)) \leq 4.$$

3 Concluding Remarks

We conclude this paper with the following open problems.

Open Problem 3.1. Calculate $d_{trk}(P_n \square P_m), d_{trk}(C_n \square$ *Cm,* and *dtrk(Pn□Cm)*

Open Problem 3.2. Calculate $d_{trk}(P(n,k))$ where $P(n,k)$ is generalized Peterson graph.

Open Problem 3.3. If G be any graph with given δ and Δ then $\delta \leq d_{trk}(G) \leq \Delta$.

Open Problem 3.4. For which classes of graphs is $d_{rk}(G) = d_{trk}(G)$ for every graph G of a class?

The extended abstract of the article is presented in CTW 2018: 16th Cologne-Twente Workshop (CTW18) on graphs and Combinatorial Optimization, Paris.

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