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# A Note on Ricci Solitons with Some Potential Vector Field

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**Abstract:** ***The present paper deals with the study of some classes of Riemannian manifolds whose metric is Ricci soliton with concircular potential vector field and solenoidal vector field. If the manifold is super quasi-Einstein, then some interesting results are obtained.***

**Keywords:** Ricci soliton, concircular potential vector field, concircular transformation, quasi-Einstein manifold, semi-symmetric connection

## 1. Introduction

In 1982, Hamilton [13] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([15], [16]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold $(M, g)$defined as follows:

$$\frac{\partial}{\partial t}g_{ij}(t) = -2\, R_{ij}$$

where R is the Ricci tensor.

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton $(g, V, \lambda)$ on a Riemannian manifold $(M,g)$ is a generalization of an Einstein metric such that [14]

$$\pounds_V g + 2S + 2\lambda g = 0 \qquad (1.1)$$

Where $\pounds_V$ is the Lie derivative operator along the vector field $V$ on $M$ and $\lambda$ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda$ is negative, zero and positive respectively.

If $\lambda \in C^\infty(M)$ then the metric satisfying (1.1) is called Ricci almost soliton. In this connection it is mentioned that Recently Hui and Patra studied Ricci al- most solution on Riemannian manifolds.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the longstanding Poincare conjecture posed in1904. There after Ricci solitons on Riemannian manifolds have been studied by various authors such as Bejanand Crasmareanu [1], Chen and Deshmukh [6], Deshmukh et al. [8], and many others.

A vector field $U$ on a Riemannian manifold $(M^n,g)$ is called concircular vector field [12], if

$$\nabla_X U = \mu X$$

For any vector field $X$ tangent to $M$, where $\nabla$ is the Riemannian connection and $\mu$ is a non-trivial function on $M$. Concircular vector fields also known as geodesic fields in literature since integral curves of such vector fields are geodesics. Recently Chen [5] studied Ricci solitons with concircular potential vector field. A vector field $U$ on a Riemannian manifold $(M^n,g)$ is called solenoidal vector field if $divU = 0$, where 'div' denotes the divergence.

Motivated by the above studies, the present paper deals with the study of Ricci solitons on some classes of Riemannian manifolds $(M^n,g)$ with concircular potential vector field and solenoidal vector field. The paper is organized as follows. Section 2 is concerned with some preliminaries. Section 3 is devoted to the study of Ricci solitons on flat and concircularly flat Riemannian manifolds $(M^n,g)$ with concircular potential vector field and solenoidal vector field. In section 4, we study Ricci solitons on super quasi-Einstein manifolds with concircular potential vector field and solenoidal vector field.

## 2. Preliminaries

This section deals with some preliminaries, which will be required in the sequel. Let $(g, V, \lambda)$ be a Ricci soliton on a Riemannian manifold $(M^n,g)$ with concircular potential vector field $V$. Then we have from (1.2) that $\nabla_X U = \mu X$ which implies that

$$(\pounds_V g)(Y,Z) = 2\mu g(Y,Z) \qquad (2.1)$$

For any vector fields $Y$, $Z$ Using (2.1) in(1.1) we get (2.2)

$$S(Y,Z) = -(\lambda+\mu) g(Y,Z).$$

In this case it is proved that [5] $\mu$ must be a non-zero constant say $b$. Thus, we have

$$S(Y,Z) = -(\lambda+b) g(Y,Z) \qquad (2.3)$$

Let $\{e_i,\ i=1,2,\cdots,n\}$ be an orthonormal frame field at any point of the manifold $(M^n, g)$. Then

$$\sum_{i=1}^{n} \pounds_V g(e_i, e_i) = 2div\, V$$

For any vector field V.

## 3. Flat and Concircularly flat Riemannian manifolds

This section deals with the study of Ricci solitons on Riemannian manifolds with concircular potential vector field.

Let $(M^n,g)$ be a Riemannian manifold, which is flat. Then $R(X, Y)Z=0$ and hence

$$S(Y, Z) = 0 \quad (3.1)$$

where $R$ and $S$ are respectively the curvature tensor and Ricci tensor. From (2.3) and (3.1), we get $\lambda= —b<0$ or$>0$ according as $b>0$ or$<0$. Also $\lambda$ /=0 as $b$ /=0.

This leads to the following:

**Theorem 3.1.** *Let $(g, V, \lambda)$ be a Ricci soliton with concircular potential vector field V on a flat Riemannian manifold $(M^n, g)$. Then$(g, V, \lambda)$ is never be steady and it is shrinking and expanding according as $b > 0$ or $<0$.*

Using (3.1) and (1.1), we get

$$(£_V g)(Y,Z) + 2\lambda g(Y,Z) = 0 \quad (3.2)$$

Putting $Y=Z=e_i$ in (3.2) and taking summation over $i,1\leq i\leq n$, we get (3.3)

$$divV + n\lambda = 0. \quad (3.3)$$

If $V$ is solenoidal then $divV= 0$ and hence from (3.3), we get $\lambda = 0$, i.e. the Ricci soliton is steady.

This leads to the following:

**Theorem 3.2.** *Let $(g, V, \lambda)$ be a Ricci soliton with solenoidal vector field V on a flat Riemannian manifold $(M^n, g)$. Then $(g, V, \lambda)$ is always steady.*

A concircular curvature tensor is an interesting invariant of a concircular transformation. A transformation of a Riemannian manifold $M$, which trans- forms every geodesic circle of $M$ into a geodesic circle, is called a concircular transformation [17]. Here geodesic circle means a curve in $M$ whose first curvature is constant and whose second curvature is identically zero. Thus, the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. The interesting invariant of a concircular transformation is the concircular curvature Tensor $\tilde{C}$, which is defined by [17]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(X,Y)Z - g(X,Z)Y] \quad (3,4)$$

Where $R$ is the curvature tensor and $r$ is the scalar curvature of the manifold. Also Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

A Riemannian manifold $(M^n,g)$ is said to be concircularly flat if $\tilde{C}(X,Y)Z= 0$, i.e.

$$R(X,Y)Z - \frac{r}{n(n-1)}[g(X,Y)Z - g(X,Z)Y] = 0 \quad (3,5)$$

Contracting above equation we get

$$S(Y,Z) = \frac{r}{n}g(Y,Z) \quad (3,6)$$

From (2.3) and (3.6), we get $n\lambda= -r + bn$

$$-\frac{r}{n} + b < 0 = 0 > 0.$$

**Theorem 3.3.** *Let $(g,V,\lambda)$ be a Ricci soliton with concircular potential vector field V on a concircularly flat Riemannian manifold $(M^n,g)$. Then $(g,V,\lambda)$ shrinking, steady and expanding according as $-\frac{r}{n} + b < 0 = 0$ or $>$ 0.*

In view of (3.6), (1.1) yields

$$(£_V g)(Y,Z) + 2\left(\lambda + \frac{r}{n}\right) g(Y,Z) = 0 \quad (3.7)$$

Putting $Y=Z=e_i$ in (3.7) and taking summation over$i$, $1\leq i\leq n$, we get

$$div\, V + n\lambda + r = 0 \quad (3.8)$$

If $V$ is solenoidal then $divV$=0 and hence from (3.8), we get $\lambda = -\frac{r}{n}$

Hence we can state the following:

**Theorem3.4.** *Let $(g,V,\lambda)$ be a Ricci soliton with solenoidal vector field Von a concircularly flat Riemannian manifold $(M^n,g)$. Then $(g,V,\lambda)$ is shrinking, steady and expanding depending upon the sign of scalar curvature.*

## 4. Super quasi-Einstein manifolds

Let $(M^n,g)$ (n≥3) be a Riemannian manifold. Let $U_s = \{x\epsilon M, S \neq \frac{r}{n} g \text{ at } x\}$

Then the manifold $(M^n,g)$ is said to be quasi-Einstein manifold ([4],[7],[9], [10],[11]) if on $U_S \subset M$, we have

$$S - \alpha g = \beta\, A \otimes A \quad (4.1)$$

Where $A$ is a unit 1-form on $U_S$ and $\alpha$, $\beta$ are some functions on $U_S$. It is clear that the1-form $A$ as well as the function $\beta$ are non-zero at every point on $U_S$. From the above definition, it follows that every Einstein manifold is quasi-Einstein. In particular, every Ricci-flat manifold (e.g. Schwarzschild spacetime) is quasi-Einstein. The scalars $\alpha$, $\beta$ are known as the associated scalars of the manifold. Also, the unit 1-form $A$ is called the associated 1-form of the manifold defined by $g(X, \rho)=A(X)$ for any vector field $X$; $\rho$

being a unit vector field, called the generator of the manifold. Such an $n$-dimensional quasi-Einstein manifold is denoted by $(QE)_n$.

As a generalization of quasi-Einstein manifold, Chaki [2] introduced the notion of generalized quasi-Einstein manifolds. A Riemannian manifold $(M^n,g)$ $(n>2)$ is said to be generalized quasi-Einstein manifold if its Ricci tensor $S$of type (0, 2) is not identically zero and satisfies the following:

$S(X,Y)=\alpha g(X,Y)+\beta A(X)A(Y)+\gamma[A(X)B(Y)+A(Y)B(X)]$, (4.2)

Where $\alpha$, $\beta$, $\gamma$ are scalars of which $\beta/=0$, $\gamma/=0$, $A$, $B$ are non-zero 1-forms such that $g(X,\rho)=A(X)$, $g(X,\mu)=B(X)$ for all $X$ and $\rho$, $\mu$ are two unit vector fields mutually orthogonal to each other. In such a case $\alpha$, $\beta$ and $\gamma$ are called the associated scalars, $A$, $B$ are called the associated 1-forms and $\rho$, $\mu$ are the generators of the manifold. Such an $n$-dimensional manifold is denoted by $G(QE)_n$.

In [3], Chaki also introduced the notion of super quasi-Einstein manifold. A Riemannian manifold $(M^n,g)$, $(n>2)$ is called super quasi-Einstein manifold if its Ricci tensor $S$ of type (0,2) is not identically zero and satisfies the following:

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X) + \delta D(X,Y) \qquad (4.3)$$

here $\alpha$, $\beta$, $\gamma$, $\delta$ are non-zero scalars, $A$, $B$ are two non-zero 1forms such that $g(X,\rho)=A(X)$, $g(X,\mu)=B(X)$ for all vector fields

$X$ and $\rho$, $\mu$ are unit vectors such that $\rho$ is perpendicular to $\mu$ and $D$ is a symmetric (0,2) tensor with zero trace, which satisfies the condition $D(X,\rho)=0$ for all vector Fields $X$ .Here $\alpha$, $\beta$, $\gamma$ and $\delta$ are called the associated scalars, $A$,$B$ are the associated 1-forms of the manifold. Such an $n$-dimensional manifold is denoted by $S(QE)_n$.

Setting $X=Y=e_i$ in (4.3) and taking summation over$i$,$1\le i\le n$, we obtain

$$r = n\alpha + \beta + \gamma \qquad (4.4)$$

where $r$ is the scalar curvature of the manifold. From (2.3) and (4.3), we get

$$(\gamma + b + \alpha)g(X,Y) + \beta A(XA(Y) + \gamma[A(X)B(Y + A(Y)B(X) + \delta D(X,Y) = 0 \qquad (4.5)$$

Putting $X=Y=e_i$ in (4.5) and taking summation over $i$, $1\le i\le n$, we obtain

$$n(\lambda + b + \alpha) + \beta = 0 \qquad (4.6)$$

Which implies by virtue of (4.4) that $\lambda=—(b+\alpha)$, which is <0, =0 or>0

$b+\alpha>0=$ or <0

This leads to the following:

**Theorem 4.1.** *Let $(g,V,\lambda)$ be a Ricci soliton with concircular potential vector field V on a super quasi-Einstein manifold (respectively, generalized quasi- Einstein manifold, quasi-Einsteinmanifold) $(M^n,g)$. Then $(g,V,\lambda)$ is shrinking, steady and expanding according as* $b + \alpha > 0, = 0$ *or* $< 0$.

For any vector field $V$.

In view of (4.3), (1.1) yields

$$(£_V g)(Y,Z) + 2(\lambda + \alpha)g(Y,Z) + 2\beta A(Y)A(Z) + 2\gamma[A(Y)B(Z) + A(Z)B(Y)] + 2\delta D(Y,Z) = 0 \qquad (4.7)$$

Putting $Y= Z= e_i$ in (4.7) and taking summation over $i$, $1\le i\le n$, we obtain

$$divV + n(\lambda + \alpha) + \beta = 0 \qquad (4.8)$$

Using (4.4) in (4.8) we get

$$divV + r + n\lambda = 0 \qquad (4.9)$$

If $V$ is solenoidal then $divV=0$ and hence from (4.9), we get $\lambda = -\frac{r}{n}$.

Hence, we can state the following:

**Theorem4.2.** *Let $(g,V,\lambda)$ be a Ricci soliton with solenoidal vector field V on a super quasi-Einstein manifold respectively, generalized quasi-Einstein manifold, quasi-Einstein manifold) $(M^n,g)$. Then $(g,V,\lambda)$ is shrinking, steady and expanding depending upon the sign of scalar curvature*

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