International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Complex Plane and Cartesian Plane

Understanding C and R^2

B. Tabsum

Assistant Professor, Department of Mathematics, Government College of Arts, Science and Commerce Quepem Goa, India

Abstract: ***This study has been undertaken to understand the similarities and the differences between $\mathbb{C}$ and $\mathbb{R}^2$. Here we study the properties of complex plane and Cartesian planes in detail and compare them.***

Keywords: field

1. Introduction

Complex numbers are the extension of real numbers and are defined be the set $\mathbb{C} = \{a + ib : a, b \in \mathbb{R},\ i = \sqrt{-1}\}$, any complex number can be denoted by $z = a + ib$

i is not a real number

Proof

Firstly, let's note that i is a solution of the equation $x^2 + 1 = 0$

$i.e. x^2 + 1 = 0$

$\Rightarrow\ x^2 = -1\ \Rightarrow x = \pm\sqrt{-1}$

$if\ x\ is\ a\ real\ number\ , then\ by\ law\ of\ tricotomy\ x > 0\ or\ x < 0\ or\ \ x = 0$

$i > 0\ \Rightarrow \sqrt{-1} > 0 \Rightarrow \left(\sqrt{-1}\right)^2 > 0 \Rightarrow -1 > 0 \Rightarrow\Leftarrow$

$i < 0\ \Rightarrow \sqrt{-1} < 0 \Rightarrow \left(\sqrt{-1}\right)^2 > 0 \Rightarrow -1 > 0 \Rightarrow\Leftarrow$

$i > 0\ \Rightarrow \sqrt{-1} = 0 \Rightarrow \left(\sqrt{-1}\right)^2 = 0 \Rightarrow -1 = 0 \Rightarrow\Leftarrow$

Therefore in any case i fails to be a real numer

$\mathbb{R}^2 = \{(a, b): a, b \in \mathbb{R}\}$

Similarities between $\mathbb{C}$ and $\mathbb{R}^2$

- $\mathbb{C}$ and $\mathbb{R}^2$ have same cardinality
  Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ defined by $f(a, b) = a + ib$, clearly f is a bijection for
  $f(a_1, b_1) = f(a_2, b_2) \Rightarrow a_1 + ib_1 = a_2 + ib_2 \Rightarrow a_1 = a_2$ and $b_1 = b_2 \Rightarrow (a_1, b_1) = (a_2, b_2)$
  Hence f is one-one. Also for any $c + id$, $f(c, d) = c + id$
  Thus f is a bijection and hence the result.
- $\mathbb{C}$ and $\mathbb{R}^2$ both are groups with respect to addition

  We prove the statement for $\mathbb{C}$ fiirst
  Let $z_1, z_2, z_3 \in \mathbb{C}$, where $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2\ and\ z_3 = a_3 + ib_3$ and $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$
  $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2) \in \mathbb{C}$ ⋯ **closure property**

$$
\begin{aligned}
(z_1 + z_2) + z_3 &= [(a_1 + ib_1) + (a_2 + ib_2)] + (a_3 + ib_3) \\
&= [(a_1 + a_2) + i(b_1 + b_2)] + (a_3 + ib_3) \\
&= ((a_1 + a_2) + a_3) + i((b_1 + b_2) + b_3) \\
&= \left(a_1 + (a_2 + a_3)\right) + i\left(b_1 + (b_2 + b_3)\right) \\
&= a_1 + ib_1 + \left((a_2 + a_3) + i(b_2 + b_3)\right) \\
&= a_1 + ib_1 + [(a_2 + ib_2) + (a_3 + ib_3)] \\
&= z_1 + (z_2 + z_3)
\end{aligned}
$$
⋯**associativity property**

Let $w = u + iv$

Suppose $z + w = z \Rightarrow (a + ib) + (u + iv) = a + ib$

$$
\begin{aligned}
&\Rightarrow (a + u) + i(b + v) = a + ib \\
&\Rightarrow a + u = a\ and\ b + v = b \\
&\Rightarrow u = 0\ and\ v = 0
\end{aligned}
$$

Similarly, $w + z = z \Rightarrow (u + iv) + (a + ib) = a + ib$

$$
\begin{aligned}
&\Rightarrow (u + a) + i(v + b) = a + ib \\
&\Rightarrow u + a = a\ and\ v + b = b \\
&\Rightarrow u = 0\ and\ v = 0
\end{aligned}
$$

Thus for any $z \in \mathbb{C}$ there exists a unique $w = 0 + 0i = 0$ such that $z + w = z = w + z$ ⋯ **identity property**

Let $W = U + iV$

Suppose $z + W = 0 \Rightarrow (a + ib) + (U + iV) = 0 + i0$

$$
\begin{aligned}
&\Rightarrow (a + U) + i(b + V) = 0 + i0 \\
&\Rightarrow a + U = 0\ and\ b + V = 0 \\
&\Rightarrow U = -a\ and\ V = -b
\end{aligned}
$$

Similarly, $W + z = 0 \Rightarrow (U + iV) + (a + ib) = 0 + i0$

$$
\begin{aligned}
&\Rightarrow (U + a) + i(V + b) = 0 + i0 \\
&\Rightarrow U + a = 0\ and\ V + b = 0 \\
&\Rightarrow U = -a\ and\ V = -b
\end{aligned}
$$

Thus for any $z \in \mathbb{C}$ there is a $W = -a + (-b)i = -z$ such that $z + W = 0 = W + z$ ⋯ **additive inverse property**

Now we prove that $\mathbb{R}^2$ is a group with respect to addition

Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \mathbb{R}^2$

$(a_1, b_1) + (a_2, b_2) = ((a_1 + a_2), (b_1 + b_2)) \in \mathbb{R}^2$ ⋯ **closure property**

$$
\begin{aligned}
(z_1 + z_2) + z_3 &= [(a_1, b_1) + (a_2, b_2)] + (a_3, b_3) \\
&= [(a_1 + a_2), (b_1 + b_2)] + (a_3, b_3) \\
&= \left((a_1 + a_2) + a_3\right), \left((b_1 + b_2) + b_3\right) \\
&= \left(a_1 + (a_2 + a_3)\right), \left(b_1 + (b_2 + b_3)\right) \\
&= (a_1, b_1) + \left((a_2 + a_3), (b_2 + b_3)\right) \\
&= (a_1, b_1) + [(a_2, b_2) + (a_3, b_3)]
\end{aligned}
$$

⋯ **associativity property**

Let $(u, v) \in \mathbb{R}^2$

Suppose $(a, b) + (u, v) = (a, b)$

$$
\begin{aligned}
&\Rightarrow \left((a + u), (b + v)\right) = (a, b) \\
&\Rightarrow a + u = a\ and\ b + v = b \\
&\Rightarrow u = 0\ and\ v = 0
\end{aligned}
$$

Similarly, $\Rightarrow (u, v) + (a, b) = (a, b)$

$$\Rightarrow (u + a) + i(v + b) = (a, b)$$
$$\Rightarrow u + a = a \text{ and } v + b = b$$
$$\Rightarrow u = 0 \text{ and } v = 0$$

Thus for any $z \in \mathbb{C}$ there exists a unique (0,0) such that $(a, b) + (u, v) = (a, b) = (u, v) + (a, b)$ ⋯ **identity property**

Let $(U, V) \in \mathbb{R}^2$

Suppose $(a, b) + (U, V) = (0,0)$

$$\Rightarrow \big((a + U), (b + V)\big) = (0,0)$$
$$\Rightarrow a + U = 0 \text{ and } b + V = 0$$
$$\Rightarrow U = -a \text{ and } V = -b$$

Similarly, $\Rightarrow (U, V) + (a, b) = (0,0)$

$$\Rightarrow \big((U + a), (V + b)\big) = (0,0)$$
$$\Rightarrow U + a = 0 \text{ and } V + b = 0$$
$$\Rightarrow U = -a \text{ and } V = -b$$

Thus for any $z \in \mathbb{C}$ there is a $(-a, (-b))$ such that $(a, b) + (U, V) = (0,0) = (U, V) + (a, b)$ ⋯ **additive inverse property**

Difference between $\mathbb{C}$ and $\mathbb{R}^2$

- You can add a real number to a complex number but you cannot add a real number to an element in $\mathbb{R}^2$
  $let\ z = 3 + 5i\ , then\ z + 4 = 3 + 5i + 4 = 7 + 5i$
  $where\ as\ for\ (3,5) \in \mathbb{R}^2, (3,5) + 4\ is\ not\ defined$
- Multiplying two complex numbers gives a complex number, where as any two numbers in $\mathbb{R}^2$ can be treated as vectors that leads to either Dot product or Cross product
  $let\ z_1 = a + ib\ and\ z_2 = c + id\ then\ z_1 z_2 = ac - bd + i(ad + bc)$
  $where\ as, if\ u = (a, b)\ and\ v = (c, d)\ in\ \mathbb{R}^2, then\ u \cdot v = ac + bd$
  $and\ u \times v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = (ad - bc)\hat{k}\ a\ unit\ vector$ in the direction perpendicular to $x - axis\ and\ y - axis$
- A complex number is a scalar where as an element in $\mathbb{R}^2$ is referred to as a vector
- We can divide two complex numbers but division of two vectors is not defined
- Elements of R are called as vectors where as that of $\mathbb{C}$ are scalars
- C is a field, whereas, R^2 is just a vector space

Acknowledgment

I thank Springer verlag and McGraw hill International publications And I thank the references mentioned.

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