Fuzzy Probability and Its Application

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Abstract: ***This paper deals with a Fixed Reorder Quantity System with Backorder where the demand follows uniform distribution. The concept of fuzzy probability is applied here where the single objective stochastic inventory model is discussed and the numerical results are illustrated.***

Keywords: Fuzzy Optimization, Fuzzy Probability, Uniform Distribution, Fuzzy Non-linear Programming, Multi-objective Non-linear Programming.

1. Introduction

Stochastic programming has been developed in various directions like chance constrained programming, recourse programming, and multi-objective stochastic programming etc. Fuzzy mathematical programming also involves uncertainty due to the presence of ambiguous data in the problem. Starting from Zimmermann (1983), Fuzzy mathematical programming has been developed by many researchers. However there are some real world situations, where randomness and fuzziness come together in a decision-making problem. For example, if the probability (Ram is young) is low, then the random variable representing "young" is a fuzzy random variable. The decision-making problems having such type of ambiguous information are known as Fuzzy Stochastic Programming problem. Many researchers like Liu and Iwamura (1998), Luhandjula (1983, 2003), Luhandjula and Gupta (1996) have derived different methods to solve such type of decision-making problems. All these techniques convert the Fuzzy stochastic problems to its deterministic equivalent, following the concept of deterministic equivalent of Stochastic Programming and the deterministic equivalent of Fuzzy programming as introduced by Zimmermann (1983). Seliaman and Rahman (2008) discussed Optimizing inventory decisions in a multi-stage supply chain under stochastic demands. Buckley (2003) introduced a new approach and applications of fuzzy probability and after that Buckley & Eslami (2003, 2004) contributed three remarkable articles about uncertain probabilities. Hwang & Yao (1996) discussed the independent fuzzy random variables and their applications. Formalization of fuzzy random variables are considered by Colubi, Dominguez-Menchero, Lopez-Diaz & Ralescu (2001). Kratschmer (2001) analyzed a unified approach to fuzzy random variables. Fuzzy random variables are considerd as scalar expected value operators by Liu & Liu (2003). Luhandjula (2003) discussed a mathematical programming in the presence of fuzzy quantities and random variables.

In this paper we discussed a fixed reorder quantity system with backorder and we apply fuzzy probability technique to solve the model where demand follows uniform distribution. Finally we analyze the problem numerically, also.

2. Mathematical Model

A Fixed Reorder Quantity System with Backorder

Here the policy is to order a lot size Q when the inventory level drops to a reorder point r and it is supposed that the inventory position of an item is monitored after every transaction. The demand in any given interval of time is a random variable and the expected value of demand in a unit of time, say a year, is D. We let x denote the demand during the lead-time and f (x) denote its probability distribution.

The fixed procurement cost is A and the unit variable procurement cost is C. The cost of carrying a unit of inventory for one unit of time is h. All shortages are backordered at a cost of π per unit short, regardless of the duration of the shortage. Because of the probabilistic nature of demand, the number of cycles per year is a random variable that averages D/Q. The procurement cost per cycle is A+CQ.

The shortages cost per cycle is $\pi\bar{b}(r)$, where $\bar{b}(r)$ is the expected number of shortages per cycle and is a function of reorder point r. The amount of the shortage at the end of a cycle, when the replenishment order is received, is b (x, r) = max [0, x - r], which has the expected value

$$\bar{b}(r) = \int_r^\infty (x-r)\,f(x)dx$$

μ is the expected demand during a lead time. The quantity Q/2 is often called the cycle stock and r - μ is referred to as the safety stock. Thus safety stock for the system is the amount by which the reorder point exceeds the average usage during a lead-time.

The average annual cost is:

$$K(Q,r) = \frac{AD}{Q} + CD + h\left(\frac{Q}{2} + r - \mu\right) + \frac{\pi D\bar{b}(r)}{Q} \quad \dots(1)$$

3. Fuzzy Non-linear Programming (FNLP) Technique to Solve Multi-Objective Non-Linear Programming Problem (MONLP)

A Multi-Objective Non-Linear Programming (MONLP**)** problem or Vector Minimization problem (VMP) may be taken in the following form**:**

$$\mathrm{Min}f(x) = (f_1(x), f_2(x), \dots\dots\dots, f_k(x))^T \quad \dots(2)$$

Subject to

$x \in X = \{x \in R^n : g_j(x) \leq or = or \geq b_j \ for \ j = 1,2,......,m$

and

$l_i \leq x \leq u_i (i = 1,2,....,n)$ }.

Zimmermann (1978) showed that fuzzy programming technique can be used to solve the multi-objective programming problem.

To solve MONLP problem, following steps are used**:**

Step 1: Solve the MONLP problem of equation (2) as a single objective non-linear programming problem using only one objective at a time and ignoring the others, these solutions are known as ideal solution.

Step 2: From the result of step1, determine the corresponding values for every objective at each solution derived. With the values of all objectives at each ideal solution, pay-off matrix can be formulated as follows:

$$\begin{array}{cc} & \begin{array}{cccc} f_1(x) & f_2(x) & .... & f_k(x) \end{array} \\ \begin{array}{c} x^1 \\ x^2 \\ .... \\ x^k \end{array} & \begin{bmatrix} f_1^*(x^1) & f_2(x^1) & .... & f_k(x^1) \\ f_1(x^2) & f_2^*(x^2) & .... & f_k(x^2) \\ .... & .... & .... & .... \\ f_1(x^k) & f_2(x^k) & .... & f_k^*(x^k) \end{bmatrix} \end{array}$$

Here $x^1, x^2, ..., x^k$ are the ideal solutions of the objective functions $f_1(x), f_2(x), ..., f_k(x)$ respectively.

So $U_r = \max\{f_r(x_1), f_r(x_2), ...f_r(x_k)\}$

and $L_r = \min\{f_r(x_1), f_r(x_2), ...f_r(x_k)\}$

[L_r and U_r be lower and upper bounds of the r^{th} objective functions $f_r(x)$ $r = 1,2,.....,k)$]

Step 3: Using aspiration level of each objective of the MONLP problem of equation (2) may be written as follows:

Find x so as to satisfy

$f_r(x) \tilde{\leq} L_r \ (r = 1,2,........,k)$

$x \in X$

Here objective functions of equation (2) are considered as fuzzy constraints. These types of fuzzy constraints can be quantified by eliciting a corresponding membership function:

$\mu_r(f_r(x) = 0$ or $\rightarrow 0$ if $f_r(x) \geq U_r$

$= \mu_r(f_r(x)$ if $L_r \leq f_r(x) \leq U_r \ (r = 1,2,........,k)$

$= 1$ if $f_r(x) \leq L_r$ ….(3)

Having elicited the membership functions (as in equation (4.8)) $\mu_r(f_r(x)$ for r = 1, 2, …… , k, introduce a general aggregation function

$\mu_{\tilde{D}}(x) = G(\mu_1(f_1(x)), \mu_2(f_2(x)), ..., \mu_k(f_k(x)))$.

So a fuzzy multi-objective decision making problem can be defined as

Max $\mu_{\tilde{D}}(x)$

Subjectto $x \in X$ ….(4)

Here we adopt the fuzzy decision as:

Fuzzy decision based on minimum operator (like Zimmermann's approach (1976). In this case equation (4) is known as $FNLP_M$.

Then the problem of equation (4), using the membership function as in equation (3), according to min-operator is reduced to:

Max α ….(5)

Subject to $\mu_i(f_i(x) \geq \alpha$ for $i = 1,2,........,k)$

$x \in X \quad \alpha \in [0,1]$

Step 4: Solve the equation (5) to get optimal solution.

4. Single Objective Stochastic Inventory Model [SOSIM]

Most of the probabilistic inventory models are considered as an unconstrained probabilistic optimization model. But, in real life, problems are considered under some limited restrictions, e.g. total floor space is not unlimited. Here F is the total floor space area. So the following model may be considered:

Minimize average annual cost under floor space constraints. It is a Single Objective Stochastic Inventory Model [SOSIM]

The model can be stated as:

$$\text{Min } K(Q_1, Q_2, ...Q_n, r_1, r_2, ...r_n) = \sum_{i=1}^{n} \Big(\frac{A_i D_i}{Q_i} + C_i D_i + h_i\Big(\frac{Q_i}{2} + r_i - \mu_i\Big) + \Big(\frac{\pi_i D_i}{Q_i}\Big)\bar{b}_i(r_i)\Big) \quad ...(6)$$

subject to the constraints

$$\sum_{i=1}^{n} p_i Q_i \leq F$$

$Q_i \geq 0$ (i = 1, 2, …., n).

5. Fuzzy Probability

Let $Y=\{y_1, y_2, ..., y_n\}$ be a finite set & let P be a probability function define on all subsets of Y with $P(\{y_i\})=p_i$, $1 \leq i \leq n$, $0<p_i<1$, all i, and $\sum_{i=1}^{n} p_i = 1$.We may substitute a fuzzy number $\tilde{p}_i$ for p_i, for some i ,to obtain a discrete (finite)fuzzy probability distribution.

In some problem, because of the way the problem is stated, the values of all the p_i are crisp & known. For example, in case of throwing a fair die where, p_1= probability of getting

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six and p_2 is the probability of not getting six. Clearly $p_1=1/6$ and $p_2=5/6$. But in many problems p_i are not known exactly and they are estimated either from a random sample or from "expert opinion".

Assume that we do not know the values of pi and we do not have any data to estimate their values. Then we may obtain numbers for the p_i from some group of experts. This group could consist of only one expert. He is to estimate the value of some probability p. We can solicit this estimate from the expert as it is done in estimating job times project scheduling. Let a, b, c be the respective pessimistic, most likely and optimistic values of p. we then ask the expert to give values of a, b and c and we construct the triangular fuzzy number $\tilde{p}$ =(x, y, z) for p. if we have a group of M experts all to estimate the value of p we solicit the x_i, y_i and z_i, $1\leq i\leq M$, from them. Let x, y and z be the average of x_i, y_i and z_i respectively. The simplest thing to do is to use (x, y, z) for $\tilde{p}$.

Similarly in case of continuous probability distributions several parameters that are involved in the distributions can be considered as a fuzzy number and in this manner the concept of fuzzy probability can be introduced in continuous case also.

5.1 Fuzzy Uniform Distribution

We know that in case of uniform distribution, the density function is f (x; a, b)=1/(b-a) for a<x<b and f(x; a, b)=0, otherwise. But for fuzzy uniform $\tilde{a}$ and $\tilde{b}$ are considered as triangular fuzzy number, i.e. $\tilde{a}$ and $\tilde{b}$ are the fuzzy estimator of a and b. Now using the fuzzy uniform density we wish to compute the fuzzy probability of obtaining a value in the interval [c,d]. This fuzzy probability is denoted as $\tilde{P}[c,d]$. There is uncertainty in the end points of the uniform density but there is no uncertainty in the fact that we have a uniform density. So, given any $s\varepsilon\, a[\alpha]$ and $t\varepsilon\, b[\alpha]$, s < t, we have f (x; s, t)=1/(t-s) on [s,t]and it equals to zero otherwise, for all $0\leq\alpha\leq1$. This enables us to find fuzzy probabilities. Let L(c,d; s,t) be the length of the interval [s,t] $\cap$ [c,d]. Then

$P[c,d]$[α] = {L(c,d; s,t)/(t-s): $s\varepsilon\, a[\alpha]$, $t\varepsilon\, b[\alpha]$,s < t} …(7)

for all $\alpha\varepsilon[0,1]$. Equation (7.10) defines the α-cuts and we put these α-cuts together to obtain the fuzzy set $\tilde{P}[c,d]$. To find an α-cut of $\tilde{P}[c,d]$ we find the probability of getting a value in the interval[c,d] for each uniform density f(x; s,t) for all $s\varepsilon\, a[\alpha]$ and $t\varepsilon\, b[\alpha]$, s< t.

1) Solution of Single Objective Fuzzy Stochastic Inventory Model [SOFSIM]

Here parameters of the stochastic models are fuzzy numbers. We consider the following SOFSIM:

$$MinK(Q,r)= \text{Min}\ \ K(Q_1,Q_2,.....Q_n,r_1,r_2,.....r_n) = \sum_{i=1}^{n}\left(\frac{A_iD_i}{Q_i}+C_iD_i+h_i\left(\frac{Q_i}{2}+r_i-\mu_i\right)+\left(\frac{\pi_iD_i}{Q_i}\right)\bar{b}_i(r_i)\right) \quad \text{…(8)}$$

subject to the constraints

$$\sum_{i=1}^{n}p_iQ_i\leq F$$

$Q_i\geq 0$ (i = 1, 2, …., n).

$$K_{MIN}(\alpha)=\min\sum_{i=1}^{n}\left[\frac{A_iD_i}{Q_i}+C_iD_i+h_i\left(\frac{Q_i}{2}+r_i-\mu_i\right)+\left(\frac{\pi_iD_i}{Q_i}\right)\bar{b}_i(r_i)\right]$$

$$K_{MAX}(\alpha)=\max\sum_{i=1}^{n}\left[\frac{A_iD_i}{Q_i}+C_iD_i+h_i\left(\frac{Q_i}{2}+r_i-\mu_i\right)+\left(\frac{\pi_iD_i}{Q_i}\right)\bar{b}_i(r_i)\right]$$

Here S indicates the statement, "The fuzzy parameters involved in the expression belong to their respective α-cuts".

Let, $\bar{K}(\alpha)=(\,K_{MIN}(\alpha),\ K_{MAX}(\alpha))$

To minimize the above interval,

We consider $K_{AV}(\alpha)=\dfrac{K_{MIN}(\alpha)+K_{MAX}(\alpha)}{2}$

Now the following MOSIM is formed:

Min $K_{MIN}(\alpha)$

Min $K_{AV}(\alpha)$

subject to the constraints

$$\sum_{i=1}^{n}p_iQ_i\leq F$$

$Q_i\geq 0$ (i = 1, 2, …., n). …(9)

6. Demand Follows Uniform distribution

We assume that lead time demand for the period for the i^{th} item is a random variable which follows uniform distribution and if the decision maker feels that demand values for item i below a_i or above b_i are highly unlikely and values between a_i and b_i are equally likely, then the probability density function $f_i(x)$ are given by:

$$f_i(x) = \begin{cases}\dfrac{1}{b_i-a_i} & \text{if } a_i\leq x\leq b_i\\ 0 & \text{otherwise}\end{cases} \quad \text{for i = 1, 2, …, n.}$$

So, $\bar{b}_i(s_i)=\dfrac{(b_i-s_i)^2}{2(b_i-a_i)}$ for i = 1, 2, …, n …(10)

Where, $\bar{b}_i(s_i)$ are the expected number of shortages per cycle and all these values of $\bar{b}_i(s_i)$ affects the desired models.

7. Numerical

To solve the model (8), where the demand follows fuzzy random variable, actually we have to solve the model (10) and thus the following data are considered:

In case of *Fuzzy Uniform* demand:
F = 60,000 ft^2 , f_1=3 ft^2, f_2=2 ft^2, A_1 = \$70, A_2 = \$60, h_1 = \$2, h_2 = \$1, D_1 = 80,000 unit, D_2 = 60,000 unit, a_1 = 50, b_1 = 60, a_2 = 75, b_2 = 90, $\alpha = 0.8$ $\alpha' = 1 - \alpha$ = 0.2

$\tilde{a}_1$ = (5,10,15); $\tilde{b}_1$ = (40,50,60); $a_1[\alpha']$ = [10-5α′, 10+5α′]; $b_1[\alpha']$ = [50-10α′, 50+10α′].
$\tilde{a}_2$ = (25,30,35); $\tilde{b}_2$ = (60,70,80); $a_2[\alpha']$ = [30-5α′, 30+5α′]; $b_2[\alpha']$ = [70-10α′, 70+10α′].
[All the cost related parameters are measured in "\$"]

Thus the following results are obtained in **Table - 1**:

Solution of the model (8)

Table 1

Prob. Distribution	K_{MIN} *(\$)	K_{MAX} *(\$)	K_{AVG} *(\$)	Q_1*	Q_2*	r_1*	r_2*	Aspiration Level
UNIFORM	661475.9	661578.2	661590.3	16142.17	786.7516	60.068	89.994	0.7314

8. Conclusion

This paper models a fixed reorder quantity system with backorder under Uniform distribution. This model can be analyzed also in case of exponential and normal demand and corresponding results can be illustrated numerically.

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