International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

# Group {1, −1, *i*, −*i*} Cordial Labeling of Some Shadow Graphs

**M. K. Karthik Chidambaram[1], S. Athisayanathan[2], R. Ponraj[3]**

[1]Department of Mathematics, St. Xavier's College, Palayamkottai 627 002, Tamil Nadu, India
Email: *karthikmat5[at]gmail.com*

[2]Department of Mathematics, St. Xavier's College, Palayamkottai 627 002, Tamil Nadu, India
Email: *athisxc[at]gmail.com*

[3]Department of Mathematics, Sri Paramakalyani College, Alwarkuruchi 627 412, Tamil Nadu, India
Email: *ponrajmath[at]gmail.com*

**Abstract:** ***Let G be a (p,q) graph and A be a group. Let f: V (G) → A be a function. The order of u ∈ A is the least positive integer n such that $u^n$ = e. We denote the order of u by o(u). For each edge uv assign the label 1 if (o(f (u)), o(f (v))) = 1 or 0 otherwise. The function f is called a group A Cordial labeling if |vf (a) − vf (b)| ≤ 1 and |ef (0) − ef (1)| ≤ 1, where vf (x) and ef(n) respectively denote the number of vertices labeled with an element x and number of edges labeled with n(n=0, 1). A graph which admits a group A Cordial labeling is called a group A Cordial graph. The Shadow graph $D_2$(G) of a connected graph G is constructed by taking two copies of G, G′ and G" and joining each vertex u′ in G′ to the neighbours of the corresponding vertex u" in G". In this paper we define group {1, −1, i, −i} Cordial graphs and prove that the Shadow graphs of Path $P_n$ and Cycle $C_n$ are group {1, −1, i, −i} Cordial. We also characterize shadow graph of Complete graph $K_n$ that are group {1, −1, i, −i} Cordial.***

**Keywords:** Cordial labeling, group A Cordial labeling, group {1, −1, *i*, −*i*} Cordial labeling, Shadow graph.

**AMS subject classification:** 05C78

## 1. Introduction

Graphs considered here are finite, undirected and simple. Let A be a group.

The order of $a \in A$ is the least positive integer $n$ such that $a^n = e$. We denote the order of $a$ by $o(a)$. Cahit [3] introduced the concept of Cordial labeling. Motivated by this, we define group A cordial labeling and investigate some of its properties. We also define group {1, −1, *i*, −*i*} cordial labeling and discuss that labeling for some standard graphs [1, 2] . The Shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$, $G'$ and $G$" and joining each vertex $u'$ in $G'$ to the neighbours of the corresponding vertex $u$" in $G$". In this paper we define group {1, −1, *i*, −*i*}

Cordial graphs and prove that the Shadow graphs of Path $P_n$ and Cycle $C_n$ are group {1, −1, *i*, −*i*} Cordial. We also characterize shadow graph of Complete graph $K_n$ that are group {1, 1, *i*, *i*} Cordial. Terms not defined here are used in the sense of Harary [5] and Gallian [4].

The greatest common divisor of two integers $m$ and $n$ is denoted by $(m, n)$ and $m$ and $n$ are said to be *relatively prime* if $(m, n) = 1$. For any real number $x$, we denote by $\lfloor x \rfloor$, the greatest integer smaller than or equal to $x$ and by $\lceil x \rceil$, we mean the smallest integer greater than or equal to $x$.

A *path* is an alternating sequence of vertices and edges, $v_1$, $e_1$, $v_2$, $e_2$, ..., $e_{n-1}$, $v_n$, which are distinct, such that $e_i$ is an edge joining $v_i$ and $v_{i+1}$ for $1 \le i \le n-1$. A path on $n$ vertices is denoted by $P_n$. A path $v_1$, $e_1$, $v_2$, $e_2$, ..., $e_{n-1}$, $v_n$, $e_n$, $v_1$ is called a cycle and a cycle on $n$ vertices is denoted by $C_n$. If $G$ is a graph on $n$ vertices in which every vertex is adjacent to every other vertex, then $G$ is called a complete graph and is denoted by $K_n$.

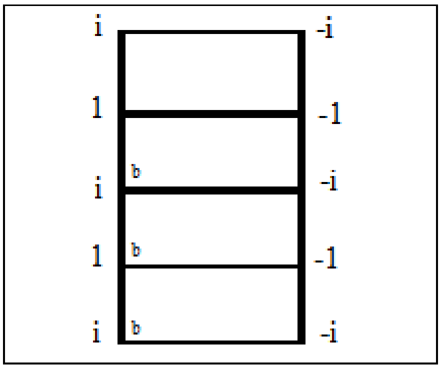


**Figure 2.1**

## 2. Group {1, −1, i, −i} Cordial graphs

**Definition 2.1.** Let G be a (p,q) graph and consider the group $A = \{1, -1, i, -i\}$ with multiplication. Let $f : V(G) \to A$ be a function. For each edge $uv$ assign the label 1 if $(o(f(u)), o(f(v))) = 1$or 0 otherwise. The function $f$ is called a group {1, −1, *i*, −*i*} Cordial labeling if $|vf(a) - vf(b)| \le 1$ and $|ef(0) - ef(1)| \le 1$, where $vf(x)$ and $ef(n)$ respectively denote the number of vertices labeled with an element x and number of edges with n(n=0,1). A graph which admits a group {1, −1, *i*, −*i*} Cordial labeling is called a group {1, −1,

$i, -i\}$ Cordial graph.

**Example 2.2:** A simple example of a group $\{1, -1, i, -i\}$ given in Fig. 2.1.

Cordial graph is

**Definition 2.3**. *The Shadow graph D2(G) of a connected graph G is constructed by taking two copies of G, G′ and G" and joining each vertex u′ in G′ to the neighbours of the corresponding vertex u" in G".*

**Example 2.4.** *The Shadow graph* $D_2(P_3)$ *is given in Fig.* 2.2.

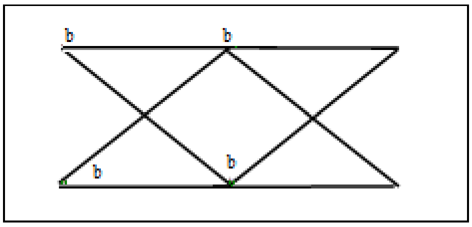

**Figure 2.2**

**Remark 2.5.** *If G is a* $(p, q)$ *graph then* $D_2(G)$ *is a* $(2p, 4q)$ *graph.*

**Theorem 2.6.** *For every* $n > 1$, $D_2(P_n)$ *is group* $\{1, -1, i, -i\}$ *cordial.*

**Proof.** Let $u_1, u_2, \ldots, u_n$ be the vertices of one copy of $P_n$ and $v_1, v_2, , v_n$ be the corresponding vertices of the other copy of $P_n$. Number of vertices in $D_2(P_n)$ is $2n$ and number of edges is $4(n-1)$. Let $f$ be a labeling defined on $V(D_2(P_n))$ as follows:

$$\text{Define } f(u_j) = \begin{cases} 1 & \text{if j is even,} \\ -1 & \text{if j is odd} \end{cases} \quad \text{and}$$

$$f(v_j) = \begin{cases} i & \text{if j is even,} \\ -i & \text{if j is odd} \end{cases}$$

If $n$ is even, say $n = 2k$, then number of edges receiving label 1 is $(k-1)4+2 = 4k-2$. If $n$ is odd, say $n = 2k + 1$, then number of edges induced with label 1 is $4k$. Table 1 shows that $f$ is a group $\{1, -1, i, -i\}$ cordial labeling.

**Theorem 2.7**. *For every n ,* $D_2(C_n)$ *is group* $\{1, -1, i, -i\}$ *cordial.*

**Proof.** Let $u_1, u_2, \ldots, u_n$ be the vertices of one copy of $C_n$ and $v_1, v_2, , v_n$ be the vertices of the other copy of $C_n$. Number of vertices in $D_2(C_n)$ is $2n$ and number of edges in $D_2(C_n)$ is $4n$.

**Table 1**

| n | vf (1) | vf (−1) | vf (i) | vf (−i) | ef (0) | ef (1) |
|---|---|---|---|---|---|---|
| 2k, k ≥ 1, k ε Z | k | k | k | k | 4k − 2 | 4k − 2 |
| 2k + 1, k ≥ 1, k εZ | k | k + 1 | k | k + 1 | 4k | 4k |

**Case (1):** $n$ is even.

Let $n = 2k$, $k \geq 1$, $k \in Z$. Let $f$ be a labeling defined on the vertices as follows:

$$\text{Define } f(u_j) = \begin{cases} 1 & \text{if j is even,} \\ -1 & \text{if j is odd} \end{cases} \quad \text{and}$$

$$f(v_j) = \begin{cases} i & \text{if j is even,} \\ -i & \text{if j is odd} \end{cases}$$

Number of edges with label 1 = $4k$.

**Case (2):** $n$ is odd.

Let $n = 2k + 1$, $k \geq 1$, $k \in Z$. Let $f$ be a labeling defined on the vertices as follows:

$$\text{Define } f(u_j) = \begin{cases} 1 & \text{if j is even,} \\ -1 & \text{if j is odd} \end{cases}$$

$$f(v_3) = 1 \quad \text{and}$$

$$f(v_j) = \begin{cases} i & \text{if j is even,} \\ -i & \text{if j is odd and } j \neq 3 \end{cases}$$

Number of edges with label 1 = 4k + 2.

**Table 2:** Shows that f is a group {1, −1, i, −i} cordial labeling

| n | vf (1) | vf (−1) | vf (i) | vf (−i) | ef (0) | ef (1) |
|---|---|---|---|---|---|---|
| 2k, k ≥ 1, k ε Z | k | k | k | k | 4k | 4k |
| 2k + 1, k ≥ 1, k ε Z | k + 1 | k + 1 | k | k | 4k + 2 | 4k + 2 |

**Theorem 2.8.** $D_2(K_n)$ *is group* $\{1, -1, i, -i\}$ *cordial iff* $n$ = 4, 7 *or* 9.

**Proof**. Let $u_1, u_2, \ldots, u_n$ be the vertices of one copy of $K_n$ and let $v_1, v_2, , v_n$ be the vertices of another copy of $K_n$. Note that $D_2(K_n) = K_{2n} - X$ where $X$ is a perfect matching. Number of vertices in $D_2(K_n)$ is $2n$ and number of edges is $4\, nC_2 = 4n(n-1)/2 = 2n(n-1)$. Suppose $D_2(K_n)$ *is group* $\{1, -1, i, -i\}$ *cordial* and let $f$ be a group $\{1, -1, i, -i\}$ cordial labeling of $D_2(K_n)$. So each vertex label appears n/2 times and each edge label appears $n(n-1)$ times.

**Case (1):** $n$ is even.

Let $n = 2k$, $k \geq 1$, $k \in Z$. Degree of each vertex is $2(2k-1)$. We need to give label 1 to $k$ vertices and get $2k(2k-1)$ edges with label 1. But it is possible to take only 2 vertices so that each choice gives edge label 1 to $2(2k-1)$ edges. Thus $k = 2$ and so $n = 4$.

**Case (2):** $n$ is odd.

Let $n = 2k + 1$, $k \geq 1$, $k \in Z$. Degree of each vertex is $4k$. If we give label 1 to $k$ vertices then at most $k.4k = 4k^2$ edges can receive label 1.But we need to give label 1 to $k(4k + 2) = 4k^2 + 2k$ vertices. So $k + 1$ vertices have to be given label 1. If the choice of $k + 1$ vertices is among $\{u_1, u_2, , u_n\}$ or among $\{v_1, v_2, \ldots, v_n\}$, then we have $4k + (4k-1) + \ldots + (4k-k) = 2k(2k+1)$ which on simplification gives $k^2 - 3k = 0$. So $k = 3$ and hence $n = 7$.

Suppose some vertices in the choice of $k + 1$ vertices is among $\{u1, u2, ....., un\}$ and some are in $\{v1, v2, ....., vn\}$. As $u_i$ and $v_i$ are alone non-adjacent, a group$\{1, -1, i, -i\}$ cordial labeling is possible only by taking $u_i$ and $v_i$ in pairs. Any other choice produces less edges with label 1.

**Subcase (1):** $k + 1$ is even.
We need to have $2(4k) + 2(4k - 2) + ..... + 2(4k - 2(\frac{k+1}{} - 1)) = 2k(2k + 1)$ which on simplification gives k2-4k-1=0 on simplification gives $k^2$-4k -1=0 and so k=2± √5 which isisimpossis impossible as k is an integer.

**Subcase(2):** $k + 1$ is odd.
In this case, we need to have, $2(4k) + 2(4k - 2) + ... + 2(4k - 2(\frac{k}{} - 1)) + (4k - 2k/2) = 2k(2k + 1)$ which on simplification gives $k^2 - 4k = 0$. So $k = 4$ and hence $n = 9$. Thus $n = 4$, 7 or 9.

Conversely, for $n = 4$, the function $f$ defined by $f(u1) = f(v1) = 1$, $f(u2) = f(v2) = -1$, $f(u3) = f(v3) = i$, $f(u4) = f(v4) = -i$ is a group $\{1, -1, i, -i\}$ cordial labeling. For $n = 7$, the function $g$ defined by, $g(uj) = 1$ for $1 \leq j \leq 4$, $g(uj) = -1$ for $5 \leq j \leq 7$, $g(vj) = i$ for $1 \leq j \leq 4$, $g(vj) = -i$ for $5 \leq j \leq 7$ is a group $\{1, -1, i, -i\}$ cordial labeling. For $n = 9$, the function $h$ defined by, $h(u1) = h(v1) = h(u2) = h(v2) = h(v3) = 1$, $h(uj) = -1$ for $4 \leq j \leq 8$, $h(u9) = h(v3) = h(v4) = h(v5) = i$, $h(vj) = -i$ for $6 \leq j \leq 9$ is a group $\{1, -1, i, -i\}$ cordial labeling.

**Theorem 2.9**. *$D2(Jn)$ is group $\{1, -1, i, -i\}$ cordial if and only if $n \in \{2, 4, 5, 7, 9, 11, 12, 13, 15, 17, 19\}$.*

**Proof.** $D2(Jn)$ has $2n + 8$ vertices and $4(2n + 5)$ edges. Let the vertices of ne copy of $D2(Jn)$ be labeled as $u, v, x, y, ui(1 \leq$ I of the other copy be labeled as u', v', x',y',$v_i$ $(1 \leq i \leq n)$.

**Case (1):** $n$ is even.

Let $n = 2k$, $k \geq 1$, $k \in Z$. Let $f$ be a group $\{1, -1, i, -i\}$ cordial labeling of $D2(Jn)$. Each vertex label should appear $k + 2$ times and each edge label should appear $8k + 10$ times. Any choice of $k + 2$ vertices without vertices from $\{u, v, u', v'\}$ or with more than 1 vertex from $\{u, v, u', v'\}$ is not a group {1,-1,i ,-i } cordial labeling. Without loss of generality choose $u$ and let
$f(u) = 1$. This induces $4k + 4$ edges with label 1. If $f(x) = f(x') = f(y) = f(y') = 1$, then we need to have $5 + 5 + 4 + 4 + 3(k-3) = 4k+6$ which on simplification gives $k = 6$. If $f(x) = f(x') = f(y') = 1$, then we need to have $5 + 5 + 4 + 3(2) = 4k + 6$ which gives $k = 2$. If $f(x) = f(x') = 1$, then $5 + 5 + (k - 1)3 = 4k + 6$ which gives $k = 1$. If $f(x) = f(y) = 1$, then $5 + 4 + (k - 1)3 = 4k + 6$ which gives $k = 0$. All other possibilities give negative values for $k$. Thus $k = 1, 2, 6$ so that $n = 2, 4, 12$.

**Case (2):** $n$ is odd.
Let $n = 2k + 1$, $k \geq 1$, $k \in Z$. Let $f$ be a group $\{1, -1, i, -i\}$ cordial labeling of $D2(Jn)$. Two vertex labels should appear $k + 2$ times and two other labels should appear $k + 3$ times. Number of edge labels with label 1 is $8k + 9$. As in Case(1), we need to have $f(u) = 1$. This induces edge label 1 to $4k + 6$ edges. If $f(x) = f(x') = f(y) = f(y') = 1$, then either $5+5+4+4+(k-3)3 = 4k+3$ or $5 + 5 + 4 + 4 + (k - 2)3 = 4k + 3$ and so $k = 6$ or 9. If $f(x) = f(x') = f(y) = 1$, then either $k = 5$ or $k = 8$. If $f(x) = f(x') = 1$, then $k = 4$ or $k = 7$. If $f(x) = f(y) = 1$, then either $k = 3$ or $k = 6$. If $f(x) = 1$, then $k = 2$ or $k = 5$. If $f(x) \neq 1$, $f(y) \neq 1$, $f(x') \neq 1$, $f(y') \neq 1$, then $k = 0$ or $k = 3$. Thus $k \in \{2, 3, 4, 5, 6, 7, 8, 9\}$ and so $n = 5, 7, 9, 11, 13, 15, 17, 19$.

Conversely, suppose $n$ {2, 4, 12, 5, 7, 9, 11, 13, 15, 17, 19}. If $n = 2$, define $f$ by $f(u) \stackrel{2}{=} f(x) = f(x') = 1$; $f(v) = f(y) = f(u') = -1$; $f(y') = f(v') = f(u1) = i$; $f(u2) = f(v1) = f(v2) = -i$.

If $n = 4$, define $f$ by $f(u) = f(x) = f(x') = f(y) = 1$; $f(v) = f(y') = f(u') = f(v') = -1$; $f(ui)(1 \leq i \leq 4) = i$; $f(vi)(1 \leq i \leq 4) = -i$.

If $n = 12$, define $f$ by $f(u) = f(x) = f(x') = f(y) = f(y') = 1$; $f(ui)(1 \leq i \leq 3) = 1$; $f(v) = f(u') = f(v') = f(ui)(4 \leq \leq 12)$; $f(vi) = -i$ for $1 \leq i \leq 8$. $i \leq 8) = -1$; $f(ui) = f(vi) = i(9 \leq i$

If $n = 5$,define $f$ by $f(u) = f(x) = f(ui)(1 \leq i \leq 2) = 1$; $f(v) = f(y) = f(ui)(3 \leq i \leq 4) = -1$; $f(u') = f(v') = f(x') = f(y') = f(u5) = i$; $f(vj)(1 \leq j \leq 5) = -i$.

If $n = 7$, define $f$ by $f(u) = f(x) = f(y) = f(u1) = f(u2) = 1$; $f(v) = f(u3) = f(u4) = f(u5) = f(u6) = -1$; $f(u7) = f(u') = f(v') = f(x') = f(y') = f(v1) = i$; $f(vj)(2 \leq j \leq 7) = -i$.

If $n = 9$, define $f$ by $f(u) = f(x) = f(x') = f(u_i)$ $(1 \leq i \leq 3) = 1$; $f(v) = f(y) = f(y') = f(u') = f(v'6) = f(u_4) = -1$; $f(u_j)(5 \leq j \leq 9) = f(v_1) = f(v_2) = i$; $f(vj)(3 \leq j \leq 9) = -i$.

If $n = 11$, define $f$ by $f(u) = f(x) = f(uj)(1 \leq j \leq 6) = 1$; $f(v) = f(y) = f(uj)(7 \leq j \leq 11) = f(x') = -1$; $f(y') = f(u') = f(v') = f(vj)(1 \leq j \leq 4) =$
$i$; $f(vj)(5 \leq j \leq 11) = -i$.

If $n = 13$, define $f$ by $f(u) = f(x) = f(x') = f(y) = f(y') = f(uj)(1 \leq j \leq 3) = 1$; $f(v) = f(u') = f(v') = f(uj)(4 \leq j \leq 8) = -1$; $f(uj)(9 \leq j \leq 13) = f(vj)(1 \leq j \leq 4) = i$; $f(vj)(5 \leq j \leq 13) = -i$.

If $n = 15$, define $f$ by $f(u) = f(x) = f(x') = f(uj)(1 \leq j \leq 7)) = 1$; $f(v) = f(y) = f(y') = f(u') = f(v') = f(uj)(8 \leq j \leq 12) = -1$; $f(uj)(13 \leq j \leq 15) = f(vj)(1 \leq j \leq 6) = i$; $f(vj)(7 \leq j \leq 15) = -i$.

If $n = 17$, define $f$ by $f(u) = f(x) = f(x') = f(y) = f(uj)(1 \leq j \leq 7) = 1$; $f(v) = f(y') = f(u') = f(v') = f(uj)(8 \leq j \leq 14) = -1$; $f(uj)(15 \leq j \leq 17) = f(vj)(1 \leq j \leq 7) = i$; $f(vj)(8 \leq j \leq 17) = -i$.

If $n = 19$, define $f$ by $f(u) = f(x) = f(x') = f(y) = f(y') = f(uj)(1 \leq j \leq 7) = 1$; $f(v) = f(u') = f(v') = f(uj)(8 \leq j \leq 16) = -1$; $f(uj)(17 \leq j \leq 19) = f(vj)(1 \leq j \leq 8) = i$; $f(vj)(9 \leq j \leq 19) = -i$.

In all the cases, the $f$ defined is a group $\{1, -1, i, -i\}$ cordial labeling.

## References

[1] Athisayanathan, S., Ponraj, R. and Karthik Chidambaram, M., K., Group A cordial labeling of Graphs, International Journal of Applied Mathematical Sciences, Vol.10, No.1, 2017,pp 1-11 .
[2] Athisayanathan, S., Ponraj, R. and Karthik Chidambaram, M., K., Group{1,-1,i,-i} Cordial labeling of sum of $P_n$ and $K_n$, Journal of Mathematical and computational Science, Vol.7, No.2(2017),335-346
[3] Cahit, I., cordial graphs: a weaker version of graceful and harmonious graphs, Ars Combin. 23(1987) 201-207
[4] Gallian, J. A, A Dynamic survey of Graph Labeling, The Electronic Jour nal of Combinatories *Dec*7(2015), *No.D*56.
[5] Harary, F., Graph Theory, Addison Wesley, Reading Mass, 1972.