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Comprehensive Exploration of the Archimedean Property: Insights and Proofs for Undergraduate Mathematics Students

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Abstract: ***This study has been undertaken to understand The Archimedean property in detail. Various references have been made to go in depth of the concept. An attempt is made to combine all possible results on The Archimedean property so as to help the reader to have a better insight of it. The aim of this paper is to explore maximum number of statements with proofs that can help the undergraduate students studying Mathematics to understand this property in better way.***

Keywords: field, greatest integer function, supremum, least upper bound property

1. Introduction

The Archimedean property is named after the ancient Greek mathematician Archimedes of Syracuse. One of the fundamental properties of the real numbers is the Archimedean Property, an axiom introduced by Archimedes in his work on geometry. In modern language, it says that every real number is bounded above by a natural number: For any $x \in \mathbb{R}$there is a $n \in \mathbb{N}$ so that$x \leq n$. It follows that there are no infinitesimally large or small elements in $\mathbb{R}$. That means every element in $\mathbb{R}$ is bounded. The property states that given two positive numbers x and y, there is an integer n such that $nx > y$.

For example

a) let $x = 7$ and $y = 100$, then we find that if we take $n = 15$, then $nx = 15 \times 7 = 105 > 100 = y$
b) let $x = 4$ and $y = 179$, then we find that if we take $n = 45$, then $nx = 4 \times 45 = 180 > 179 = y$
c) let $x = 100$ and $y = 7$, then we find that if we take $n = 1$, then $nx = 1 \times 100 = 100 > 7 = y$
d) let $x = 179$ and $y = 4$, then we find that if we take $n = 1$, then $nx = 1 \times 179 = 179 > 4 = y$

How do we get this n?

Clearly, $nx > y \Rightarrow n > \frac{y}{x} \Rightarrow n = \left[\frac{y}{x}\right] + 1$, where $[x]$ is the greatest integer less than or equal to x

If $x > y$, then $\left[\frac{y}{x}\right] = 0 \Rightarrow n = 0 + 1 = 1$

It also means that the set of natural numbers is not bounded above. Roughly speaking, it is the property of having no infinitely large or infinitely small elements.The Archimedean property is one of the most distinctive and useful features of the field of real numbers. This property is the result of the supremum property. The Supremum property (or the Least upper bound property) states that

$$\text{“}\boldsymbol{Every\ non-empty\ subset\ S\ of\ \mathbb{R}\ which\ is\ bounded\ above\ has\ a\ supremum\ in\ \mathbb{R}.}\text{”}$$

2. Archimedean Property In Calculus

1) The subset $\mathbb{N}$ of Natural numbers is not bounded above in$\mathbb{R}$

Proof:

Suppose $\mathbb{N}$ is bounded above, also $\mathbb{N} \subseteq \mathbb{R}$

$\Rightarrow$ $\mathbb{N}$ has the Supremum property i.e. $\mathbb{N}$ has the Least upper bound

$\Rightarrow \exists M \in \mathbb{R}$such that $n \leq M\ \forall n \in \mathbb{N}$

By the definition of supremum, for $1 > 0, \exists m \in \mathbb{N}$ such that $m > M - 1$

$\Rightarrow m + 1 > M \ \Rightarrow\Leftarrow$ (as $m + 1 \in \mathbb{N}$)

2) If $x \in \mathbb{R}$, then there is $n \in \mathbb{N}$ such that $x < n$

Proof:

Suppose $x \geq n\ \forall n \in \mathbb{N}$

$\Rightarrow$ $\mathbb{N}$ is bounded above by x

$\Rightarrow\Leftarrow$ ($\mathbb{N}$ is not bounded above)

3) Let $x, y > 0$, then there exists $n \in \mathbb{N}$ such that $x < ny$

Proof :

Since $x > 0$ and $y > 0$, $x/y > 0$

$\Rightarrow \exists\ n \in \mathbb{N}$ such that $x/y < n$

4) Let $y > 0$, then there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y$

Proof:

$y > 0\ and\ 1 > 0 \Rightarrow \exists\ n \in \mathbb{N}$ such that $1 < ny$

$$\Rightarrow \frac{1}{n} < y$$

$\because n \in \mathbb{N}, n > 0 \ \Rightarrow \frac{1}{n} > 0 \Rightarrow 0 < \frac{1}{n} < y$

5) Let $x > 0$, then there exists $n \in \mathbb{N}$ such that $n - 1 \leq x < n$

Proof :

Consider the set $S = \{k \in \mathbb{N} | x < k\}$

Claim $S \neq \emptyset$

As $x > 0$, by Archimedean property, $\exists\ m \in \mathbb{N}$ such that $x < m$

$\Rightarrow m \in S \ \Rightarrow S \neq \emptyset$

$\Rightarrow S$ has a least element (say n) i.e. $x < n$

Claim $x \geq n - 1$

Suppose $x < n - 1$

$$\Rightarrow n - 1 \in S$$

$\Rightarrow\Leftarrow$ (as n is the least element of S and $n - 1 < n$)

$\therefore n - 1 \leq x < n$

6) There exists a positive real number x such that $\mathbf{x^2 = 2}$

Proof:

Consider the set $S = \{t \in \mathbb{R} | t \geq 0 \; and \; t^2 < 2\}$

Claim $S \neq \emptyset$

As $1^2 = 1 < 2 \Rightarrow 1 \in S \Rightarrow S \neq \emptyset$

Claim S is bounded above by 2

Let $t \in S$ such that $t > 2, \Rightarrow t^2 > 4 \Rightarrow\Leftarrow$

$\therefore t < 2 \; \forall \; t \in S$, hence the claim

As $S \subseteq \mathbb{R}$ and is bounder above by 2, S has the least upper bound property

$\Rightarrow \exists \; x \in \mathbb{R}$ such that $x = \sup S$

Clearly, $x \neq 0$ as $1 \in S$ and $1 > 0$ (x is supposed to be the upper bound of S)

$\Rightarrow x > 0$

Claim $x^2 = 2$

Suppose $x^2 > 2 \Rightarrow x^2 - 2 > 0 \Rightarrow \frac{x^2-2}{2x} > 0 \; (as \; x > 0, 2x > 0)$

$\Rightarrow$ by Archimedean property, $\exists \; m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{x^2-2}{2x}$

Consider $\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2}$

$$> x^2 - \frac{2x}{m}$$

$> x^2 - (x^2 - 2) = 2$ (as $\frac{1}{m} < \frac{x^2-2}{2x} \Rightarrow \frac{2x}{m} < x^2 - 2 \Rightarrow -\frac{2x}{m} > -(x^2 - 2)$)

$\Rightarrow \left(x - \frac{1}{m}\right)^2 > 2 \Rightarrow x - \frac{1}{m} \notin S$

For any $t \in S, t^2 < 2 < \left(x - \frac{1}{m}\right)^2$

$\Rightarrow t < \left(x - \frac{1}{m}\right)$ (why? $t \geq 0$ and $x - \frac{1}{m} > 0$.. here $x > 1$ as x is supposed to be the upper bound)

As t is arbitrary, $t < \left(x - \frac{1}{m}\right) \; \forall t \in S$

$\Rightarrow \left(x - \frac{1}{m}\right)$ is an upper bound of S

$\Rightarrow\Leftarrow$ (As $x - \frac{1}{m} < x$ and x is supposed to be the Least upper bound)

Suppose $x^2 < 2 \Rightarrow 2 - x^2 > 0 \Rightarrow \frac{2-x^2}{2x+1} > 0 \; (as \; x > 0, 2x + 1 > 0)$

$\Rightarrow$ by Archimedean property, $\exists \; n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{2-x^2}{2x+1}$

Consider $\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$

$$= x^2 + \frac{1}{n}\left(2x + \frac{1}{n}\right)$$

$$\leq x^2 + \frac{1}{n}(2x + 1)(\because n \geq 1, \frac{1}{n} \leq 1)$$

$< x^2 + 2 - x^2 = 2$ (as $\frac{1}{n} < \frac{2-x^2}{2x+1} \Rightarrow \frac{2x+1}{n} < 2 - x^2$)

$\Rightarrow \left(x + \frac{1}{n}\right)^2 < 2 \Rightarrow \left(x + \frac{1}{n}\right) \in S \Rightarrow\Leftarrow$ (as $x + \frac{1}{n} > x$ and $x = \sup S$)

7) The Density theorem (Rational numbers are dense in $\mathbb{R}$)

If $x, y \in \mathbb{R}$ with $x < y$, then $\exists \; r \in \mathbb{Q}$ such that $x < r < y$. r can always be selected to be non-zero.

Proof:

Case (i) Suppose $x > 0$,

$\Rightarrow y - x > 0 \Rightarrow \exists \; n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$ (by Archimedean property)

$\Rightarrow 1 < n(y - x)$

$\Rightarrow 1 < ny - nx \cdots (1)$

Also $nx > 0 \; (as \; x > 0 \; and \; n \in \mathbb{N})$

$\Rightarrow \exists \; m \in \mathbb{N}$ such that $m - 1 \leq nx < m$ (by Archimedean property, by result 5)$\cdots$ (2)

$\Rightarrow m \leq nx + 1 < m + 1$

$$\Rightarrow m \leq nx + 1 < ny \cdots (from \; (1)) \cdots (3)$$

$\Rightarrow nx < m < ny \; \cdots \left(from \; (2) and \; (3)\right)$

$\Rightarrow x < \frac{m}{n} < y \Rightarrow x < r < m, where \; r = \frac{m}{n} \in \mathbb{Q}$

$clearly, r \neq 0 \; as \; m \neq 0$

Case (ii) Suppose $x < 0 \; and \; y > 0$

$if \; y > 0, then \; \frac{y}{2} > 0 \; and \; \frac{y}{2} < y$

$x < 0 \Rightarrow x < \frac{y}{2} < y$

From the previous case, $\exists \; r \in \mathbb{Q}, r \neq 0$, such that $\frac{y}{2} < r < y$ (by case (i))

$\Rightarrow x < r < y$

Case (iii) Suppose $x < 0 \; and \; y = 0$

Then $-x > 0 \Rightarrow \frac{-x}{2} > 0 \; and \; \frac{-x}{2} < -x$

$\Rightarrow 0 < \frac{-x}{2} < -x$

$\Rightarrow y < \frac{-x}{2} < -x \cdots (as \; y = 0)$

$\Rightarrow \exists \; r \in \mathbb{Q}, r \neq 0$ such that $\frac{-x}{2} < r < -x$

$\Rightarrow y < r < -x$

$\Rightarrow x < -r < -y$

$\Rightarrow x < s < y \; \cdots (where \; s = -r \; and \; \; y = 0 = -y)$

Clearly $s \neq 0 \; as \; r \neq 0 \; and \; s \in \mathbb{Q}$

Case (iv) Suppose $x < 0 \; and \; y < 0$

$\Rightarrow -x > 0 \; and - y > 0$

$and \; x < y \Rightarrow -y < -x$

By case (i), $\exists \; r \in \mathbb{Q}, r \neq 0 \; and - y < r < -x$

$\Rightarrow x < -r < y$

$\Rightarrow x < s < y \; (where \; s = -r, \therefore s \neq 0 \; also \; s \in \mathbb{Q})$

8) If x and y are real numbers with $\mathbf{x < y}$, then there exists an irrational number z such that $\mathbf{x < z < y}$.

Proof:

$x < y \Rightarrow \frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$

From result 8, $\exists \; r \in \mathbb{Q}, r \neq 0$ such that $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$

$$\Rightarrow x < r\sqrt{2} < y$$

$\Rightarrow x < s < y$

where $s = r\sqrt{2}$is an irrational number

9) If $\mathbf{0 \leq x < \frac{1}{n}}$ for all natural numbers, then $\mathbf{x = 0}$.

Proof:

Suppose $x \neq 0$
$\Rightarrow x > 0$
$\Rightarrow \exists\, m \in \mathbb{N}$ such that $\frac{1}{m} < x$ (by Archimedean property)
$\Rightarrow\Leftarrow$

10) If a and b are two real numbers such that a$< b + \frac{1}{n}$, $\forall$n $\in \mathbb{N}$, then a$\leq$ b

Proof:

Suppose $a \nleq b$

$$\Rightarrow a > b \Rightarrow a - b > 0$$

$\Rightarrow \exists\, m \in \mathbb{N}$ such that $\frac{1}{m} < a - b$ (by Archimedean property)

$$\Rightarrow \; a > b + \frac{1}{m} \Rightarrow\Leftarrow$$

11) If x $\in \mathbb{R}$, then there are integers p and q such that p $< x < q$

Proof:

We use Archimedean property to prove the above statement
we know that for any $x \in \mathbb{R}, \exists\, n \in \mathbb{N}$ such that $x < n$
similarly, for $-x \in \mathbb{R}, \exists\, m \in \mathbb{N}$ such that $-x < m$
$\Rightarrow x > -m$
let $p = -m$ and $q = n$,
$\Rightarrow p < x < q$

12) If x > 0, then there is m $\in \mathbb{N}$ such that $\frac{1}{m} < x < m$

Proof:

We know that $\exists\, p, q \in \mathbb{N}$ such that $\frac{1}{p} < x < q$ (by Archimedean property and by result (4))
if $p < q \Rightarrow \frac{1}{q} < \frac{1}{p} \Rightarrow \frac{1}{q} < \frac{1}{p} < x < q$

$$\Rightarrow \frac{1}{q} < x < q$$

Put $q = m$
if $p > q \Rightarrow \frac{1}{p} < x < q < p$

$$\Rightarrow \frac{1}{p} < x < p$$

Put $p = m$

13) Between two rational numbers, there is always a rational number

The proof of result 7 applies here. Any two rational numbers are also real numbers and hence there always exists a rational number between the two.

14) Between two rational numbers, there are infinitely many rational numbers

Proof:

Suppose there are finitely many rational numbers between x and y, where $x, y \in \mathbb{Q}$
Let the rational numbers be $r_1, r_2, r_3, \dots, r_n$.
Note that every rational number is a real number
And thus between any two $r_i, r_{i+1} \exists\, r \in \mathbb{Q}$ (from result 7)
$\Rightarrow$ there are $n + 1$ rational numbers between x and $y \Rightarrow\Leftarrow$

15) Between any two distinct irrational numbers, there are infinitely many irrational numbers

Proof:

Suppose there are finitely many irrational numbers betweenany two distinct irrational numbers$x$ and y
Let the irrational numbers be $r_1, r_2, r_3, \dots, r_n$.
Note that every irrational number is a real number
And thus between any two $r_i, r_{i+1} \exists\, r \in \mathbb{R} \backslash \mathbb{Q}$ (from result 8)
$\Rightarrow$ there are $n + 1$ irrational numbers between x and $y \Rightarrow\Leftarrow$

16) Between any two distinct irrational numbers, there are infinitely many rational numbers

Proof:

Suppose there are finitely many rational numbers between any two distinct irrational numbers$x$ and y
Let the rational numbers be $r_1, r_2, r_3, \dots, r_n$.
Note that every rational number is a real number
And thus between any two $r_i, r_{i+1} \exists\, r \in \mathbb{Q}$ (from result 7)
$\Rightarrow$ there are $n + 1$ rational numbers between x and $y \Rightarrow\Leftarrow$

17) Between any two distinct rational numbers, there are infinitely many irrational numbers

Proof:

Suppose there are finitely many irrational numbers between any two distinct rational numbers$x$ and y
Let the irrational numbers be $r_1, r_2, r_3, \dots, r_n$.
Note that every irrational number is a real number
And thus between any two $r_i, r_{i+1} \exists\, r \in \mathbb{R} \backslash \mathbb{Q}$ (from result 8)
$\Rightarrow$ there are $n + 1$ irrational numbers between x and $y \Rightarrow\Leftarrow$

18) If y > 0, then $\exists$ n $\in \mathbb{N}$ such that $\frac{1}{2^n} < y$

Proof: By Archimedean property, $\exists\, n \in \mathbb{N}$ such that$\frac{1}{n} < y$
Note that $2^n > n \;\; \forall\, n \in \mathbb{N}$

$$\Rightarrow \frac{1}{2^n} < \frac{1}{n} \Rightarrow \frac{1}{2^n} < \frac{1}{n} < y \Rightarrow \frac{1}{2^n} < y$$

19) 0 is the infimum of the set $\left\{\frac{1}{n} \;:\; n \in \mathbb{N}\right\}$

Proof: clearly, $\frac{1}{n} > 0 \;\forall\, n \in \mathbb{N} \Rightarrow 0$ is a lower bound of $\left\{\frac{1}{n} \;:\; n \in \mathbb{N}\right\}$
Now we show that 0 is the greatest lower bound
Suppose ϵ is another lower bound of $\left\{\frac{1}{n} \;:\; n \in \mathbb{N}\right\}$ such that$\epsilon > 0$
Then by Archimedean property, $\exists\, m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$
$\Rightarrow\Leftarrow$ as ϵ is supposed to be the lower bound of $\left\{\frac{1}{n} \;:\; n \in \mathbb{N}\right.$

20) If x is a real number such that n$|x| < 1$ for all n $\in \mathbb{N}$, then x $= 0$

Proof:

Suppose $x \neq 0$
$\Rightarrow x > 0$ or $x < 0$
Suppose $x > 0$
$\Rightarrow |x| = x$
Given $n|x| < 1 \;\forall\, n \in \mathbb{N}$
$\Rightarrow nx < 1 \;\forall\, n \in \mathbb{N}$

$$\Rightarrow x < \frac{1}{n} \forall n \in \mathbb{N}$$

But by Archimedean property $\exists\, m \in \mathbb{N}$such that $\frac{1}{m} < x \quad (x > 0)$

$\Rightarrow\Leftarrow$

Suppose $x < 0$

$\Rightarrow |x| = -x$ also $-x > 0$

Given $n|x| < 1 \; \forall\, n \in \mathbb{N}$

$\Rightarrow n(-x) < 1 \; \forall\, n \in \mathbb{N}$

$\Rightarrow -nx < 1 \; \forall\, n \in \mathbb{N}$

$$\Rightarrow -x < \frac{1}{n} \forall n \in \mathbb{N}$$

But by Archimedean property $\exists\, m \in \mathbb{N}$such that $\frac{1}{m} < -x \quad (-x > 0)$

$\Rightarrow\Leftarrow$

21) For every $\varepsilon > 0, \exists m \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon \forall n \geq m$

Proof:

We know that for any $\varepsilon > 0, \exists\, m \epsilon \mathbb{N}$such that $\frac{1}{m} < \varepsilon$ (by Archimedean property)

Further for any $n \in \mathbb{N}, n \geq m \Rightarrow \frac{1}{n} \leq \frac{1}{m} < \varepsilon$

As n is arbitrary, $\frac{1}{n} < \varepsilon \forall n \geq m$

22) There is no rational number x such that $x^2 = p$, where p is a Prime number

Proof:

Suppose there is a rational number $\frac{m}{n}$, such that $\left(\frac{m}{n}\right)^2 = p$, $n \in \mathbb{N}$ *and* $m \in \mathbb{Z}$

clearly $m \neq 0$ otherwise $\frac{m}{n} = 0$ *and* $p = 0 \Rightarrow\Leftarrow$ (as p is a prime number)

$\therefore \frac{m}{n} \neq 0$, let m and n be relatively prime

Then m^2 and n^2 are also relatively prime

And $\frac{m^2}{n^2} = \left(\frac{m}{n}\right)^2 = p \Rightarrow m^2 = pn^2$

$\Rightarrow p$ dividesm^2($as\ n^2$does not dividem^2)

$\Rightarrow p$ divides m

$\Rightarrow m = ps$ for some $s \in \mathbb{Z}$

$\Rightarrow m^2 = (ps)^2 = p^2 s^2 = pn^2$

$\Rightarrow ps^2 = n^2$

$\Rightarrow p$ divides $n^2 \Rightarrow p$ divides n

$\Rightarrow m$ and n both have a common factor p

$\Rightarrow\Leftarrow$ (as m and n are assumed to be relatively prime)

23) Let's get back to our main statement with which we began with our paper

Let x and y be any two positive real numbers, then, there is an integer n such that $nx > y$

Proof

Suppose $nx \leq y \; \forall\, n \in \mathbb{N}$

Consider $A = \{nx \,|n \in \mathbb{N}, x > 0\}$

Clearly $A \subseteq \mathbb{R}$ ...(1)

also A is bounded above by y

$\Rightarrow A$ has the least upper bound (by least upper bound property of $\mathbb{R}$)

Let z be the least upper bound of A

$\Rightarrow nx \leq z \; \forall\, n \in \mathbb{N}$ and $z \leq y$

$\because x > 0$

Then $z - x < z$

Clearly $z - x$ is not an upper bound of A

$\Rightarrow \exists\, m \in \mathbb{N}$ such that $mx > z - x$

$\Rightarrow (m+1)x > z$

$\Rightarrow\Leftarrow as\ (m+1)x \in A$

3. Conclusion

The Archimedean property is a fundamental concept in mathematics that states that for any two positive real numbers a and b, there exists a positive integer n such that $na > b$. In other words, it states that no matter how large a real number is, there is always a multiple of another real number that is greater than it.The Archimedean property has numerous applications in various fields of mathematics, including analysis, number theory, and geometry.Limits and convergence: The Archimedean property is essential in the study of limits and convergence of sequences and series of real numbers. It is used, for instance, to prove that the sequence 1/n converges to zero as n approaches infinity.

Calculus: The Archimedean property is used in the definition of the real numbers, which is the foundation of calculus. It is also used in the study of derivatives and integrals. Number theory: The Archimedean property is used in the proof of various number theoretic results, such as the irrationality of pi and the existence of infinitely many prime numbers.

Geometry: The Archimedean property is used in the construction of geometric objects, such as polygons and polyhedral. It is also used in the study of the properties of circles and spheres. In summary, the Archimedean property is a fundamental concept in mathematics that has far-reaching implications and applications in many areas of mathematics.

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