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SPRINGER FIBERS OF HOOK TYPE AND SCHUBERT POINTS

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ABSTRACT. This paper is a survey article aimed at extending the work of Tymoczko and Precup in [4] and compute the Betti numbers of Springer varieties Spr_λ in type A. Specifically, we focus on the Springer varieties associated to nilpotent operators in Jordan form of type $\lambda = n-k, 1^k$ and $\lambda = \lambda_1, \lambda_1 - 1, 1^k$, $(n \geq 5, \lambda_1 \geq 3, k = n - (2\lambda_1 - 1)$) and show that the Betti numbers of Spr_λ equals the Betti numbers of a particular union of Schubert varieties by setting the redundant permutation to zero. Furthermore, we consider a special partition of generic shape $\lambda = n-k, 1^k$, $n \geq 6, k = 3$ and give a closed formula which is useful in computing the number of redundant permutations and also a closed formula that helps in determining the number of standard tableaux (of the shape $\lambda = n-k, 1^k$, $k = 3, n \geq 6$) that generate the redundant permutations.

1. INTRODUCTION

Springer variety Spr_λ is a subvariety of the full flag variety $\mathcal{F}\ell_n(\mathbb{C})$, where $\mathcal{F}\ell_n(\mathbb{C})$ is the collection of sequences $(V_i)_{i=0}^n$ of subspaces of an n-dimensional vector space V over the complex field ($\mathbb{C}$) ordered by inclusions.That is,

$$\mathcal{F}\ell_n(\mathbb{C}) = \{V_\bullet : V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n\}$$

such that $dim V_i = i$ for each i.
$\mathcal{F}\ell_n(\mathbb{C})$ can also be seen as the quotient $GL_n(\mathbb{C})/B$, where $GL_n(\mathbb{C})$ is general linear group and B is the subgroup of upper triangular matrices, called the Borel subgroup. The isolated points $wB \in G/B$ correspond to the element of Weyl group $W \cong S_n$. Where S_n is a group of permutations of n letters.Elements of the Weyl group W indexed the $B-$orbit $C_w = BwB/B$ of $\mathcal{F}\ell_n(\mathbb{C})$ called the Schubert cell. These cells are isomorphic to affine spaces $\mathbb{C}^{l(w)}$ (Where $l(w)$ is the length of w). The closure $\overline{C}_w = \bigcup_{v \leq w} C_v$ is a subvariety of $\mathcal{F}\ell_n(\mathbb{C})$ called the Schubert variety, where $\leq$ is the partial order defined on W called the Bruhat order.

The variety Spr_λ is defined as the set of flags stabilized by a nilpotent operator X, where each nilpotent operator corresponds to a partition λ of $n > 0$, determined by the sizes of Jordan blocks of X.

$$Spr_\lambda = \{V_\bullet \in \mathcal{F}\ell_n(\mathbb{C}) : XV_i \subseteq V_i,\ 1 \leq i \leq n\}$$

Springer varieties are usually called Springer fibers, since they coincide with the fiber of the Springer resolution of singularities over X.

Tymozcko and Precup in [4] associated a permutation $w_T \in S_n$ (and called it Schubert point) to each standard tableaux of shape λ. This allowed them to give an algebraic and

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combinatorial algorithm to compute Betti numbers of Spr_λ by showing that the Betti numbers of Spr_λ coincides with the Betti number of a specific union of Schubert varieties. Though the correspondence was made expressible with the aid of row strict tableaux, but with a limitation to nilpotent operators whose corresponding partition λ has at most three rows or two columns.

Tymoczko [**?**] computed the Betti numbers of Spr_λ and show that they are enumerated using row -strict tableaux of shape λ by giving a combinatorial rule which she named dimension of row strict tableaux. Aba Mbirika in [3] links the dimension of each row strict tableau to a monomial in the Gasia-Procesi basis in [2].

Though, there are well-known formulas for Poincaré polynomials of Springer fibers in all cases [1], but our aim is to adopt the approach in [4] and extend the work of Tymoczko and Precup in [4] beyond the limitation of nilpotent operators whose corresponding partition has at most three rows or two columns .
Having said this, we specifically consider a nilpotent operators with Jordan canonical form corresponding to a partitions $\lambda = (n-k, 1^k)$ and $\lambda = \lambda_1, \lambda_1 - 1, 1^k,\ (n \geq 5, \lambda_1 \geq 3, k = n - (2\lambda_1 - 1)\ )$. This, we achieve by considering the set of union of closures of Schubert points $(\bigcup \overline{C}_{w_T})$ and take the redundant permutations to zero. By redundant permutations we mean those permutations that do not correspond to any row-strict tableaux.

Corollary 1.1. *Let $X \in \mathfrak{gl}_n(\mathbb{C})$ be a nilpotent operator of Jordan types $\lambda = (n-k, 1^k)$ or $\lambda = \lambda_1, \lambda_1 - 1, 1^k,\ (n \geq 5, \lambda_1 \geq 3, k = n - (2\lambda_1 - 1)\ )$ with more than three rows and two columns. The Poincaré polynomial of the Springer variety $P(Spr_\lambda, t)$ equals the Poincaré polynomial $P(\bigcup_{wB \in Spr_\lambda}, \overline{C}_{w_T})$, having set the redundant permutation to zero. Where T is the standard tableau associated to $wB \in Spr_\lambda$ of shape λ.*

This article contains four parts. In part 2, we briefly narrate the structure of Springer varieties. Part 3 contains details of the description of Schubert points. Part four contains detailed description of our results.

2. Springer Varieties Spr_λ

This section contains some background information on Springer varieties.
Let $n > 0$ be an integer, a partition of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ such that $\sum_{i=1}^{l} \lambda_i = n$. We say a nilpotent operator X is in Jordan form of type λ, if λ is determined by the Jordan blocks of X.

Given a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, there corresponds a Young diagram of row lengths $\lambda_i,\ 1 \leq i \leq l$, which is left and top aligned. For example, the figure below shows a nilpotent operator with Jordan type $3, 2, 1$ and the associated Young diagram.

Remark 2.1. We often write $\lambda \vdash n$ to mean that λ is a partition of n. The conjugate of λ, denoted by $\lambda' = \lambda'_1, \cdots \lambda'_l$ is obtained by reflecting $\mathcal{Y}(\lambda)$ in the main diagonal.

The dimension d_λ of Spr_λ is combinatorially expressed as $d_\lambda = \sum_{i=1}^{k} \frac{\lambda'_i(\lambda'_i - 1)}{2}$, where $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_k)$ is the conjugate partition of λ. That is , λ'_i is the number of $\lambda_j \geq i$.

Springer varieties are irreducible in two opposites extremal cases. Namely: if X has only one nontrivial block, i.e. $\lambda = n$, (in other words X is regular nilpotent) then Spr_λ consists

$$\begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ & & & 0 & 1 & \\ & & & 0 & 0 & \\ & & & & & 0 \end{bmatrix} \longleftrightarrow$$

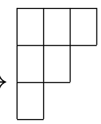

FIGURE 1. A Nilpotent operator and it's corresponding Young diagram

of the single flag $V_0 \subset V_1 \subset \cdots \subset V_n$ where $V_i = ker(X^i)$.
At the other extreme, if $\lambda = (1^n)$,then Spr_λ coincides with the whole flag variety$\mathcal{F}\ell_n(\mathbb{C})$. In any other case, Springer varieties are reducible into irreducible components. Readers are referred to [1] for details on irreducible components of Spr_λ.

3. Schubert Points of Row- Strict Tableaux of Shape $\lambda = n-k, 1^k$

A row strict tableaux τ of shape λ is a filling of a Young diagram from $[n]$ such that, numbers in the rows increase from left to right,but if the filling of such τ increases from left to right and from top to bottom, we call it standard Young tableaux which we sometimes denote by T and the set of all standard tableaux corresponding to λ as $(St)^\lambda$ We denote the set of all row strict tableaux corresponding to λ by $(rst)^\lambda$.

1	2	5
6	7	
4		
3		

,

1	2	5
6	7	
3		
4		

FIGURE 2. Row strict and Standard tableaux

For each standard tableau $T \in (St)^\lambda$ there is an associated permutation $w_T \in S_n$ called Schubert point. It known in [4] that these Schubert points are such that the length $l(w_T)$ of each w_T is equal to the dimension of the corresponding $T \in (St)^\lambda$.

Definition 3.1. Let $T \in (rst)^\lambda$ and $w_T \in S_n$ be the associated permutation, where $w_T = s_{i_1}s_{i_2}\cdots s_{i_l}$ (expressed in reduced word), we say w_T has a jump if for any two consecutive simple transpositions $s_{i_k}s_{i_j}$, $i_k - i_j \geq 2$.

Example 3.2. $w_T = s_3s_4s_5s_3s_4s_3$ has one jump which exists between the two consecutive transpositions s_5s_3 .

Definition 3.3. Let $w_T = s_{i_1}s_{i_2}\cdots s_{i_l}$ for any $T \in (rst)^\lambda$. We say w_T has a fall if for any two consecutive transpositions $s_{i_k}s_{i_j}$, $i_k - i_j = 1$.

Example 3.4. $w_T = s_3s_4s_5s_3s_4s_3$ has one fall which exists between the last two transpositions s_4s_3 .

Definition 3.5. Let $w_T = s_{i_1}s_{i_2}\cdots s_{i_l}$, we define a sub-permutation $w_{i_k+1} = s_{i_1}s_{i_2}\cdots s_{i_k}$, which is a sub-string of w_T such that there is neither a jump nor a fall between any two consecutive transpositions.

Example 3.6. Let $w_T = s_3s_4s_5s_3s_4s_3$ has three sub-permutations which are: $w_6 = s_3s_4s_5,\ w_5 = s_3s_4$ and $w_4 = s_3$

Remark 3.7. The link between the monomial associated to the dimension of T in [3] and the corresponding Schubert point is made obvious by the rule

$$w_T \longmapsto \prod_{k=2}^{n-1} x_{k+1}^{l(w_{k+1})}, \tag{3.1}$$

where $l(w_{k+1})$ is the length of sub-permutation w_{k+1}.

3.1. **Realizing row-strict tableaux from the associated Schubert points.** Here, we consider a systematic procedure of realizing a row-strict tableau from the associated Schubert point. Specifically, we consider row-strict tableaux of shape $\lambda = 3,2,1$ and highlight algorithms through which a row strict tableaux could be computed given an arbitrary Schubert point.
For any $T \in (rst)^{\lambda}$, let $T^i,\ i \leq n$ be the tableau obtained by deleting $j,\ i < j \leq n$.

Now, let T be a row-strict tableau of shape $\lambda = 3,2,1$,and $w_T = s_3s_4s_3s_2$ be a Schubert points, we realize T from w_T as follows:

- The first sub-permutation from the left is $w_5 = s_3s_4$. This implies that the number of rows above and of the same length with the row containing 5 plus the number of rows greater in length than the row containing 5 (either above or below) in T^5 must be two.Hence, 5 must occur in first box (from the left) of the third row of. That is

5		

- $w_4 = s_3$ implies that the box containing 4 in T^4 has just one row either of the same length above it or of greater length either above or below . In this case, 4 could either be in the first box of the first row, first box or second box of the second row, but since T is row-strict, 4 must be in the second box of the second row.

	4	
5		

- $w_3 = s_2$, this is obvious as 3 will fill the first box in the second row and we have

3	4	
5		

- 1,2 and 6 will fill the remaining cells accordingly. Hence we have

1	2	6
3	4	
5		

4. Outlining the main theorem

In this section, we outline the procedures that lead to our main theorem. The key step in the proof of the main theorem is to carefully list out all permutations $v \in S_n$ such that $v \leq w_T$ (in Bruhat order) for each Schubert point associated to standard tableaux of shape $\lambda = 3, 1, 1, 1$ and map the redundant permutations to zero.

Theorem 4.1. *Let T be a row-strict tableau of hook shape $\lambda = (n-k,\ 1^k),\ n > 2,\ 1 \leq k \leq n-2$ and $\lambda = \lambda_1, \lambda_1 - 1, 1^k,\ (n \geq 5, \lambda_1 \geq 3, k = n-(2\lambda_1 - 1)\ )$. For $s_{i_1}s_{i_2}s_{i_3}\cdots s_{i_{q-1}}s_{i_q} = w' \leq w_T$ such that ,$i_1 = n-1$ and $l(w') \geq \lambda_1'$, with some $s_{i_j}s_{i_{j'}}$ such that $|i_j - i_{j'}| \geq 2,\ 1 \leq j \leq q-1,\ 2 \leq j' \leq q$. Let $\#v$ be the number of sub-permutations after s_{i_j}, then there is no row strict tableau associated to w' if any of the conditions bellow is satisfied*

(1) *if $\#v \geq \lambda_1' - j$,*
(2) *if $\#v = 1$ and $l(v) \geq \lambda_1' - j$*

Proof. Given $\lambda = (n-k, 1^k),\ 1 \leq k \leq n-2$. Suppose there exists $s_{i_1}s_{i_2}s_{i_3}\cdots s_{i_{q-1}}s_{i_q} = w' \leq w_T$ with $i_1 = n-1$ and $|i_1 - i_2| \geq 2$, then n fills the cell in the second row of the Young diagram. Hence, there are k cells left in the first column to be filled. In this case, if $\#v \geq k$ then there will be no enough cells to take care of the remaining sub-permutations. Again, if $|i_2 - i_3| \geq 2$, then $i_2 + 1$ fills one of the remaining cells such that there is one cell above it, and the empty cell(s) in the column will now be $k-1$. As we move to the right of the string, the number cells for possible filling reduce. Now denote by $l(s)$ the length of the remaining string and κ the number of the remaining cells.
Suppose there is no jump or fall in the remaining string and $l(s) \geq \kappa$ then the filling(s) of the remaining cells becomes impossible.Hence w' has no corresponding row-strict tableau. If there are falls in the remaining string, such that the fall continues and $l(s) \geq \kappa$ then w' has no row strict tableau. □

Definition 4.2. Let T be a row-strict tableau of shape λ and w_T it's corresponding Schubert point. For any $w' \leq w_T$ (in Bruhat order), we call w' a redundant permutation if there exist no row-strict tableau realizable from w'.

Remark 4.3. We classify permutations w' described in the above theorem as redundant permutations.

We conquer the limitation on the computation of Betti numbers of Springer varieties in [4], by setting the redundant permutations equal to zero. This enables us to compute the Poincaré polynomial of Springer fibers associated to partition of hook shape with no limitation on either rows or columns.

Corollary 4.4. *Let $X \in \mathfrak{gl}_n(\mathbb{C})$ be a nilpotent operator of Jordan types $\lambda = (n-k, 1^k)$ or $\lambda = \lambda_1, \lambda_1 - 1, 1^k,\ (n \geq 5, \lambda_1 \geq 3, k = n-(2\lambda_1 - 1)\ )$ with more than three rows and two columns. The Poincaré polynomial of the Springer variety $P(Spr_\lambda, t)$ equals the Poincaré* *polynomial* $P(\bigcup_{wB\in Spr_\lambda},\overline{C}_{w_T})$*, having set the redundant permutation to zero. Where* T *is the standard tableau associated to* $wB \in Spr_\lambda$ *of shape* λ*.*

Proof. since the partitions λ in these cases have more than three rows or two columns, there would definitely exits some $w' \leq w_T$ with no associated row-strict tableau which we referred to as redundant permutations. Setting the redundant permutations to zero, the equality between $P(Spr_\lambda, t)$ and $P(\bigcup_{wB\in Spr_\lambda},\overline{C}_{w_T})$ is established in theorem 4.4 and theorem 3.5 of [4] □

Example 4.5. Consider $\lambda = 3, 1, 1, 1$, there are 120 row-strict tableaux, 10 out of which are standard tableaux. The 10 standard tableaux and their associated Schubert points w_T are:

$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}$ $s_3s_4s_5s_3s_4s_3$, $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}$ $s_3s_4s_5s_3s_4s_2$, $\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1} 6\\\cline{1-1}\end{array}$ $s_3s_4s_5s_2s_3s_2$, $\begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$ $s_2s_3s_4s_2s_3s_2$,

$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1} 4\\\cline{1-1} 6\\\cline{1-1}\end{array}$ $s_3s_4s_5s_2s_3s_1$, $\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2\\\cline{1-1} 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$ $s_2s_3s_4s_2s_3s_1$, $\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 6\\\cline{1-1}\end{array}$ $s_3s_4s_5s_1s_2s_1$, $\begin{array}{|c|c|c|}\hline 1&5&6\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$ $s_1s_2s_3s_1s_2s_1$,

$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}$ $s_3s_4s_5s_3s_4s_1$, $\begin{array}{|c|c|c|}\hline 1&4&6\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 5\\\cline{1-1}\end{array}$ $s_2s_3s_4s_1s_2s_1$,

Let $G = GL_6$. For each Schubert point associated to the standard tableau, we consider the union of the Shubert varieties

$$\overline{C}_{s_3s_4s_5s_3s_4s_3}, \overline{C}_{s_3s_4s_5s_3s_4s_2}, \overline{C}_{s_3s_4s_5s_2s_3s_2}, \overline{C}_{s_2s_3s_4s_2s_3s_2}, \overline{C}_{s_3s_4s_5s_2s_3s_1}, \overline{C}_{s_2s_3s_4s_2s_3s_1}, \overline{C}_{s_3s_4s_5s_1s_2s_1},$$
$$\overline{C}_{s_1s_2s_3s_1s_2s_1}, \overline{C}_{s_3s_4s_5s_3s_4s_1}, \overline{C}_{s_2s_3s_4s_1s_2s_1}.$$

That the set $\bigcup \overline{C}_{w_T}$ of all permutations $w \leq w_T$ for each $T \in (St)^\lambda$ in Bruhat order is

- $s_3s_4s_5s_3s_4s_3,\ s_3s_4s_5s_3s_4,\ s_3s_4s_5s_4s_3,\ s_4s_5s_3s_4s_3,\ s_3s_4s_5s_3,\ s_3s_4s_5s_4,\ s_5s_3s_4s_3,\ s_4s_5s_3s_4,\ s_4s_5s_4s_3,$ $s_3s_4s_5\ s_3s_4s_3\ s_5s_3s_4\ s_4s_5s_3\ s_4s_5s_4\ s_5s_4s_3\ s_3s_4,\ s_5s_3,\ s_4s_5,\ s_4s_3, s_5s_4,\ s_3,\ s_4,\ s_5,\ e$
- $s_3s_4s_5s_3s_4s_2,\ s_3s_4s_5s_3s_4,\ s_3s_4s_5s_3s_2,\ s_3s_4s_5s_4s_2,\ s_4s_5s_3s_4s_2,\ s_3s_4s_5s_3,\ s_3s_4s_5s_4,\ s_3s_4s_5s_2,$ $s_3s_4s_3s_2,\ s_5s_3s_4s_2,\ s_4s_5s_3s_4,\ s_4s_5s_3s_2,\ s_4s_5s_4s_2,\ s_4s_3s_4s_2,\ s_3s_4s_5,\ s_3s_4s_3,\ s_3s_4s_2,\ s_5s_3s_4,\ s_5s_3s_2,$ $s_4s_5s_3,\ s_4s_5s_4,\ s_4s_5s_2,\ s_4s_3s_2,\ s_5s_4s_2,\ s_3s_4,\ s_5s_3,\ s_3s_2,\ s_4s_5,\ s_4s_3,$ $s_4s_2,\ s_5s_2,\ s_2,\ s_3,\ s_4,\ s_5,\ e$
- $s_3s_4s_5s_2s_3s_2,\ s_3s_4s_5s_2s_3,\ s_3s_4s_5s_3s_2,\ s_3s_4s_2s_3s_2,\ s_4s_5s_2s_3s_2,\ s_3s_4s_5s_2,\ s_3s_4s_5s_3,\ s_3s_4s_2s_3,$ $s_3s_4s_3s_2,\ s_3s_4s_5s_2s_3s_2,\ s_5s_3s_2s_3,\ s_4s_5s_2s_3,\ s_4s_5s_3s_2,\ s_4s_2s_3s_2,\ s_3s_4s_5,\ s_3s_4s_2,$ $s_3s_4s_3,\ s_5s_3s_2,\ s_4s_5s_2,\ s_4s_5s_3,\ s_4s_2s_3\ s_4s_3s_2,\ s_5s_2s_3,\ s_3s_4,\ s_5s_3,\ s_3s_2,$ $s_4s_5, s_4s_2,\ s_4s_3,\ s_5s_2, s_2s_3,\ s_2,\ s_3,\ s_4,\ s_5,\ e$
- $s_2s_3s_4s_2s_3s_2,\ s_2s_3s_4s_2s_3,\ s_2s_3s_4s_3s_2,\ s_3s_4s_2s_3s_2,\ s_2s_3s_4s_2s_3,\ s_2s_3s_4s_3s_2,\ s_2s_3s_4s_2,\ s_2s_3s_4s_3,$ $s_4s_2s_3s_2,\ s_3s_4s_2s_3,\ s_3s_4s_3s_2,\ s_2s_3s_4\ s_2s_3s_2,\ s_4s_2s_3,\ s_3s_4s_2,\ s_3s_4s_3,\ s_4s_3s_2,\ s_2s_3,\ s_2s_4,\ s_3s_4,\ s_3s_2,$ $s_4s_3,\ s_2,\ s_3,\ s_4\ e$

- $s_3s_4s_5s_2s_3s_1,\ s_3s_4s_5s_2s_3,\ s_3s_4s_5s_2s_1,\ s_3s_4s_5s_3s_1,\ s_3s_4s_5s_3s_1,\ s_5s_3s_2s_3s_1,\ s_4s_5s_2s_3s_1,\ s_3s_4s_5s_2,$
  $s_3s_4s_5s_3,\ s_3s_4s_5s_1,\ s_3s_4s_2s_3,\ s_3s_4s_2s_3,\ s_3s_4s_2s_1,\ s_3s_4s_3s_1,\ s_5s_3s_2s_3,\ s_5s_3s_2s_1,\ s_3s_2s_3s_1,\ s_4s_5s_2s_3,$
  $s_4s_5s_2s_1,\ s_4s_5s_3s_1,\ s_4s_2s_3s_1,\ s_5s_2s_3s_1,\ s_3s_4s_5,\ s_3s_4s_2,\ s_3s_4s_3,\ s_3s_4s_1,\ s_5s_3s_2,\ s_5s_3s_1,\ s_3s_2s_1,$
  $s_4s_5s_2,\ s_4s_5s_3,\ s_4s_5s_1,\ s_4s_2s_3,\ s_4s_2s_1,\ s_4s_3s_1,\ s_5s_2s_1,\ s_2s_3s_1,\ s_3s_4,\ s_3s_2,\ s_3s_1,\ s_4s_5,$
  $s_4s_2,\ s_4s_3,\ s_4s_1,\ s_4s_1,\ s_5s_2,\ s_5s_3,\ s_5s_1,\ s_2s_1,\ s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ e$
- $s_2s_3s_4s_2s_3s_1,\ s_2s_3s_4s_2s_3,\ s_2s_3s_4s_2s_1,\ s_2s_3s_4s_3s_1\ s_3s_4s_2s_3s_1,\ s_2s_3s_4s_2,\ s_2s_3s_4s_3,\ s_2s_3s_4s_1,\ s_2s_3s_2s_1,$
  $s_4s_2s_3s_1,\ s_3s_4s_2s_3,\ s_3s_4s_2s_1,\ s_3s_4s_3s_1,$
  $s_2s_3s_4,\ s_2s_3s_2,\ s_2s_3s_1,\ s_4s_2s_3,\ s_4s_2s_1,\ s_3s_4s_2,\ s_3s_4s_3,$
  $s_3s_4s_1,\ s_3s_2s_1,\ s_4s_3s_1,\ s_2s_3,\ s_4s_2,\ s_2s_1,\ s_3s_4,\ s_3s_2,\ s_3s_1,\ s_4s_3,\ s_4s_1,\ s_1,\ s_2,\ s_3,\ s_4,\ e$
- $s_3s_4s_5s_1s_2s_1,\ s_3s_4s_5s_1s_2,\ s_3s_4s_5s_2s_1,\ s_3s_4s_1s_2s_1,\ s_5s_3s_1s_2s_1,$
  $s_4s_5s_1s_2s_1,\ s_3s_4s_5s_1,\ s_3s_4s_5s_2,$
  $s_3s_4s_5s_1,\ s_3s_4s_1s_2,\ s_3s_4s_2s_1,\ s_5s_3s_1s_2,\ s_5s_3s_2s_1,\ s_3s_1s_2s_1,\ s_4s_5s_1s_2,\ s_4s_5s_2s_1,\ s_4s_1s_2s_1,\ s_5s_1s_2s_1,$
  $s_3s_4s_1,\ s_3s_4s_2,\ s_5s_3s_1,\ s_5s_3s_2,\ s_3s_1s_2,$
  $s_3s_2s_1,\ s_4s_5s_1,$
  $s_4s_5s_2,\ s_4s_1s_2,\ s_4s_2s_1,\ s_5s_1s_2,\ s_5s_2s_1,$
  $s_1s_2s_1,\ s_3s_4,\ s_3s_5,\ s_3s_1,\ s_3s_2,\ s_4s_5,\ s_4s_1,\ s_4s_2,\ s_5s_1,$
  $s_5s_2,\ s_1s_2,\ s_2s_1,\ s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ e$
- $s_1s_2s_3s_1s_2s_1,\ s_1s_2s_3s_1s_2,\ s_1s_2s_3s_2s_1,\ s_2s_3s_1s_2s_1, s_1s_2s_1s_3,\ s_1s_2s_3s_2,\ s_3s_1s_2s_1,\ s_2s_3s_1s_2,\ s_2s_3s_2s_1,$
  $s_1s_2s_3,\ s_1s_2s_1,\ s_3s_1s_2,\ s_2s_2s_3,\ s_2s_3s_2,\ s_3s_2s_1, s_1s_2,\ s_3s_1,\ s_2s_1,\ s_2s_3,\ s_3s_2,\ s_1,\ s_2,\ s_3,\ e$
- $s_3s_4s_5s_3s_4s_1,\ s_3s_4s_5s_3s_4,\ s_3s_4s_5s_3s_1,\ s_3s_4s_5s_4s_1,\ s_4s_5s_3s_4s_1,\ s_3s_4s_5s_3,\ s_3s_4s_5s_4,\ s_3s_4s_5s_1,$
  $s_3s_4s_3s_1,\ s_5s_3s_4s_1,\ s_4s_5s_4s_1,\ s_4s_5s_3s_4,\ s_4s_5s_3s_1,$
  $s_4s_5s_4s_1,\ s_4s_3s_4s_1,\ s_3s_4s_5,\ s_3s_4s_3,\ s_3s_4s_1,\ s_5s_3s_4,$
  $s_5s_3s_1,\ s_4s_5s_3,\ s_4s_5s_4,\ s_4s_5s_1,\ s_4s_3s_1,\ s_5s_4s_1,$
  $s_5s_4,\ s_5s_1,\ s_1,\ 3s_3,\ s_4,\ s_5,\ e$
- $s_2s_3s_4s_1s_2s_1,\ s_2s_3s_4s_1s_2,\ s_2s_3s_4s_2s_1,\ s_2s_3s_1s_2s_1,\ s_3s_4s_1s_2s_1,\ s_2s_3s_4s_1,\ s_2s_3s_4s_2,\ s_2s_3s_1s_2,$
  $s_2s_3s_1s_2,\ s_2s_3s_2s_1,\ s_3s_4s_1s_2,\ s_3s_4s_2s_1,\ s_3s_1s_2s_1,\ s_4s_1s_2s_1,\ s_2s_3s_4,$
  $s_2s_3s_1,\ s_2s_3s_2,\ s_4s_2s_1,$
  $s_2s_1s_2,\ s_3s_4s_1, s_2s_1s_2,\ s_3s_4s_1,\ s_3s_4s_2,\ s_3s_1s_2,\ s_3s_2s_1,\ s_4s_1s_2,\ s_2s_3,$
  $s_4s_2,\ s_2s_1,\ s_3s_4,\ s_3s_4,\ s_3s_1,\ s_3s_2,\ s_4s_1,\ s_1s_2,\ s_1,\ s_2,\ s_3,\ s_4,\ e$

Remark 4.6. It is observed from the above example that there are 124 permutations in $\overline{C}_{w_T}$ instead of 120. The good news here is that exactly 4 $(s_5s_2s_3s_2s_1,\ s_5s_3s_2s_1,\ s_5s_3s_1s_2s_1, s_5s_3s_1s_2)$ out the 124 permutations are redundant and these are taken to zero. Therefore, the Poincaré polynomial is

$$P(\bigcup \overline{C}_{w_T}, t) = \sum_{k=1}^{6} b_k t^k = 10t^6 + 26t^5 + 35t^4 + 29t^3 + 14t^2 + 5t + 1$$

Lemma 4.7. *Given a partition λ of hook type, the minimal length of redundant permutation is λ_1'*

Proof. Let $s_{i_1}s_{i_2}s_{i_3}\cdots s_{i_{q-1}}s_{i_q} = w' \leq w_T$ be a permutation that satisfies any of the conditions in theorem 4.1. If $l(w') = \lambda_1'$, then , placing n in the second cell of the first column from above left the remaining λ_1' transpositions in the string with $\lambda_1' - 1$ cells,now suppose that there is no jump or fall in the remaining string, then the number of the available cells will not be sufficient.Also, if there are falls in the remaining string then there will be no enough cell to take the remaining numbers since each transposition corresponds to a number in $\{1, 2, 3, \cdots, n\}$ Since $\#v = \lambda'_1$. □

Finally, we consider a special partition of generic shape $\lambda = n-k, 1^k,\ n \geq 6, k = 3$ and give a closed formula which is useful in computing the number of redundant permutations and also a closed formula that helps in determining the number of standard tableaux (of the shape $\lambda = n-k, 1^k,\ k = 3, n \geq 6$) that generate the redundant permutations.

Theorem 4.8. *Let $\#(r_w)_\lambda$ and $\#(r_T)_\lambda$ respectively be the number of redundant permutations and the number of row standard Young Tableaux of shape $\lambda = n-3, 1^3,\ n \geq 6$ that generate the redundant permutations. Then,*

$$\#(r_w)_\lambda = \frac{\lambda_1^3 - 7\lambda_1 + 6}{3} \tag{4.1}$$

and

$$\#(r_T)_\lambda = \frac{\lambda_1^3 - 7\lambda_1 + 6}{6} \tag{4.2}$$

Proof. By continuous computation, starting with $n \geq 6$ and $\lambda = n-3, 1^3$, we came up with the table below.

n	6	7	8	9	10
λ	3,1,1,1	4,1,1,1	5,1,1,1	6,1,1,1	7,1,1,1
$\#(r_T)_\lambda$	2	7	16	30	50
$\#(r_w)_\lambda$	4	14	32	60	100

The sequence generated in the above table by $\#(r_T)_\lambda$ coincised with number U_n of inscribable triangles within a (n+4)-gon sharing with them its vertices but not its sides. That is,

$$U_n = \frac{(n-1)n(n+4)}{6}\ (OEISA005581). \tag{4.3}$$

Let $n = (\lambda_1 - 1)$, then the above equation gives $\#(r_T)_\lambda$. Obtaining $\#(r_w)_\lambda$ is straight forward since $\#(r_w)_\lambda = 2\#(r_T)_\lambda$ □

References

[1] Lucas Fresse. Betti numbers of springer fibers in type *a*. *Journal of Algebra*, 322(7):2566–2579, 2009.
[2] Adriano M Garsia and Claudio Procesi. On certain graded sn-modules and the q-kostka polynomials. *Advances in Mathematics*, 94(1):82–138, 1992.
[3] Aba Mbirika. A hessenberg generalization of the garsia-procesi basis for the cohomology ring of springer varieties. *the electronic journal of combinatorics*, 17(1):153, 2010.
[4] Martha Precup and Julianna Tymoczko. Springer fibers and schubert points. *arXiv preprint arXiv:1701.03502*, 2017.

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