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Weakly Commutative Mappings and Common Fixed-Point Theorem in Digital Metric Space

A. S. Saluja

Department of Mathematics, Institute for Excellence in Higher Education, Bhopal – 462016, Madhya Pradesh, India
Email: *drassaluja[at]gmail.com*

Abstract: ***In this paper, a common fixed point theorem for a pair of weakly commutative mappings in the setting of digital metric space is proved that satisfy a contractive condition in this space. Our result broaden and extend many prevailing known results in the literature.***

Keywords: Fixed point, contractive condition, Digital Metric Space, Digital Image, weakly commutative mapping

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1. Introduction

In the fixed point theory, there exist many generalities of metric space and one of them is the digital metric space introduced by Ozgur Ege and Ismet Karaca [8]. The concept of digital metric space is related to digital topology in which we study the topological and geometrical digital properties of an image. An Image is used as an object in computer graphic design and many other computer-related business works. In this type of work, a digital image is taken as a set of arranged points called pixels or voxels. In digital topology, we study these points and the adjacency relation between them. Rosenfeld [10] was the first to use digital topology as an apparatus and studied the properties of almost fixed points of a digital image. Later, Boxer [6, 7] gives the topological concept in the digital form. Based on this concept Ozgur Ege and Ismet Karaca [8] established digital metric space in 2015 and proved the "Banach Contraction Principle" and several other fixed-point results in this space. In the whole article, *DMS* illustrates digital metric space.

The study of common fixed points for different types of maps has always been a very interesting area in the theory of fixed points. Jungck [3] was the first who introduced commutative mappings to complete metric space in 1976 and by using the properties of these mapping he proved some common fixed point results. After that, many authors generalize and extend many results for commutative mapping with different contractive conditions in several ways. In 1982, Sessa [13] define weakly commutative mappings. These mappings are more general than commutative mappings that every commutative mapping is weakly commutative, but the converse may not be true. Recently, Asha Rani et al. [1] introduced weakly commutative and commutative mappings to digital metric space, and Sunjay Kumar et al. [2, 12], Sumitra Dalal [11], and Rashmi Rani [9] present some results for commutative and weakly commutative maps in *DMS*. With the motivation in this paper, a common fixedpoint theorem for a pair of weakly commutative mappings which satisfy a contractive condition in *DMS* is presented. Our result broaden and extend many prevailing known results in the literature. Before we prove our main result, the following definitions are needed.

2. Preliminaries

Let $F \subseteq \mathbb{Z}^n$, $n \in \mathbb{N}$ where $\mathbb{Z}^n$is a lattice point set in the Euclidean n - dimensional space and (F,Υ) represent a digital image, with Υ -adjacency relation between the members of F and (F,Φ,Υ) represent a *DMS*, where $(F,\Phi)$is a metric space.

Definition 2.1.[7]"Let l, n be two positive integers, where $1 \le l \le n$ and g,h are two distinct points,
$g = (g_1, g_2, \ldots\ldots g_n)$, $h = (h_1, h_2, \ldots\ldots h_n) \in \mathbb{Z}^n$.

Then the points g and h are said to beΥ_1– adjacent if there are at most l indices i such that $|g_i - h_i| = 1$ and for all other indices j,$|g_j - h_j| \neq 1$, $g_j = h_j$."

Definition 2.2. [7] Let $\kappa \in \mathbb{Z}^n$, then the set –
$N_\Upsilon(\kappa) = \{ \sigma \mid \sigma \text{ is } \Upsilon - \text{adjacent to } \kappa \}$
Represent the Υ – neighbourhood of κ for $n \in \{1, 2, 3\}$. Where $\Upsilon \in \{2,4,6,8,18,26\}$.

Definition 2.3. [7] Let δ, $\sigma \in \mathbb{Z}$ where $\delta < \sigma$, then the digital interval is -
$[\delta, \sigma]_\alpha = \{\alpha \in \mathbb{Z} \mid \delta \le \alpha \le \sigma\}$.

Definition 2.4.[8]"The digital image $(F,\Upsilon) \subseteq \mathbb{Z}^n$is called Υ–connected if and only if for every pair of different points g,$h \in F$, there is a set$\{g_0, g_1, \ldots\ldots g_s\}$of points of digital image (F,Υ),such that$g = g_0$,$h = g_s$,and $g_e$and g_{e+1} are Υ –neighbours where e =0,1,2…….. s-1."

Definition 2.5.[8]Let $K{:}F \to K$ is a function and $(F, \Upsilon_0) \subset \mathbb{Z}^{n}{}_{0}$,$(K, \Upsilon_1) \subset \mathbb{Z}^{n}{}_{1}$are two digital images. Then –

1) K is (Υ_0, Υ_1)-continuous if there existsΥ_0- connected subset$\sigma$of F, for every $K(\sigma)$,Υ_1-connected subset of K.
2) K is (Υ_0, Υ_1)- continuous if for everyΥ_0-adjacent point$\{\sigma_0, \sigma_1\}$of$F$, either $K(\sigma_0) = K(\sigma_1)$ or$K(\sigma_0)$ and $K(\sigma_1)$ are Υ_1-adjacent in K.
3) K is said to be (Υ_0, Υ_1)-isomorphism, if$K$ is (Υ_0, Υ_1)-continuous bijective and K^{-1} is (Υ_0, Υ_1)- continuous, also it is denoted by $F \cong K_{(\Upsilon_0, \Upsilon_1)}$.

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Definition 2.6. [8] Let a $(2, \Upsilon)$ continuous function $K: [0,\sigma]_z \to F$ s.t. $K(0) = \alpha$ and $K(\sigma) = \beta$. Then in the digital image (F, Υ), it is called a digital Υ - path from α to β.

Definition 2.7.[10]Let $K:(F, \Upsilon) \to (F, \Upsilon)$ be a (Υ, Υ) - continuous function on a digital image (F, Υ), then we said that the property of fixed point satisfied by the digital image (F, Υ) if for every (Υ, Υ) - continuous function $K: F \to F$ there exists $\alpha \in F$ such that $K(\alpha) = \alpha$.

Definition 2.8.[8]"Let $\{u_n\}$ is a sequence in digital metric space (F, Φ, Υ), then the sequence $\{u_n\}$ is called-

1) Cauchy sequence if and only if there exists $\varrho \in N$ such that, $\Phi(u_n, u_m) < \epsilon$, $\forall\, n, m > \varrho$.
2) Converge to a limit point $\ell \in F$ if for every $\epsilon > 0$, there exists $\varrho \in N$ such that for all $n > \varrho$, $\Phi(u_n, \Upsilon) < \epsilon$."

Theorem 2.9. [8]*"A digital metric space (F, Φ, Υ) is complete."*

Definition 2.10.[8]Let $K: (F, \Phi, \Upsilon) \to (F, \Phi, \Upsilon)$ be a self-map. Then K is called a digital contraction if, for all $u, \sigma \in F$ there exist $\tau \in [0, 1)$ such that,
$\Phi(K(u), K(\sigma)) \leq \tau\, \Phi(u, \sigma)$.

Proposition 2.11.[8]*"Every digital contraction map $K: (F, \Phi, \Upsilon) \to (F, \Phi, \Upsilon)$ is digitally Υ -continuous."*

Definition 2.12.[11]*Let J, K:* $F \to F$ are two self-mappings on (F, Φ, Υ). Then the point $\sigma \in F$ is said to be a coincidence point of J and K if $J(\sigma) = K(\sigma)$.Furthermore, if $J(\sigma) = K(\sigma) = \eta$ then η is said to be a point of coincidence for mappings J and K.

Definition 2.13. [1]Let $J, K: (F, \Phi, \Upsilon) \to (F, \Phi, \Upsilon)$ are two mappings defined on the digital metric space (F, Φ, Υ). Then these mappings are called weakly commutative mappings if
$\Phi(J(K(\sigma)), K(J(\sigma))) \leq \Phi(J(\sigma), K(\sigma))$, $\forall \sigma \in F$.

Lemma 2.14. [1] Let $\{u_n\}$ is a sequence in complete digital metric space (F, Φ, Υ),and if there exists $\rho \in (0, 1)$, such that $\Phi(u_{n+1}, u_n) \leq \Phi(u_n, u_{n-1})$ for all n then, sequence $\{u_n\}$ converges to a point in F.

3. Main Result

Theorem 3. *Let (F, Φ, Υ) represent a complete DMS, where Υ is an adjacency and Φ is a usual Euclidean metric on $\mathbb{Z}^n$. let $J, K: F \to F$ are two self-mappings such that $J(F) \subseteq K(F)$ satisfying the following,*

1) *Mapping K is continuous*
2) *Pair {J, K} is weakly commutative*
3) $\Phi(Ju, Jq) \leq \xi_1 \Phi(Ku, Kq) + \xi_2 \Phi(Ku, Ju) + \xi_3 \Phi(Kq, Jq)$, $\forall\, u,q \in F$ *where,* $\xi_1, \xi_2, \xi_3 \geq 0$ *and* $\xi_1 + \xi_2 + \xi_3 < 1$.

Then there exists a unique common fixed point in F for mappings J and K.

Proof: Let $u_0 \in F$ be an arbitrary point. Since $J(F) \subseteq K(F)$, let $u_1 \in F$ be chosen such that $J(u_0) = K(u_1)$. Continuing this procedure having chosen $u_n \in F$, we chose $u_{n+1} \in F$ such that,

$$K(u_n) = J(u_{n-1}), \qquad \forall n = 0,1,2,\ldots..(3.1)$$

Now, by inequality(3), we have –
$\Phi(Ju_n, Ju_{n-1}) = \Phi(Ku_{n+1}, Ku_n)$
$\leq \xi_1 \Phi(Ku_n, Ku_{n-1}) + \xi_2 \Phi(Ku_n, Ju_n) + \xi_3 \Phi(Ku_{n-1}, Ju_{n-1})$
$\leq \xi_1 \Phi(Ku_n, Ku_{n-1}) + \xi_2 \Phi(Ku_n, Ku_{n+1}) + \xi_3 \Phi(Ku_{n-1}, Ku_n)$

$\leq (\xi_1 + \xi_3) \Phi(Ku_n, Ku_{n-1}) + \xi_2 \Phi(Ku_n, Ku_{n+1})$

Implies that,

$\Phi(Ku_{n+1}, Ku_n) \leq \frac{\xi_1 + \xi_3}{1 - \xi_2} \Phi(Ku_n, Ku_{n-1})$

$\Phi(Ku_{n+1}, Ku_n) \leq \psi \Phi(Ku_n, Ku_{n-1})$

Where, $\psi = \frac{\xi_1 + \xi_3}{1 - \xi_2}$ and $\psi < 1$.

Hence, from Lemma (2.14), there exists $\beta \in F$, such that,

$$K(u_n) \to \beta \qquad (3.2)$$

Also, from(3.1) we have,

$$J(u_n) \to \beta \qquad (3.3)$$

Now, $\Phi(Ju_n, Ju_n) \leq \xi_1 \Phi(Ku_n, Ku_n) + \xi_2 \Phi(Ku_n, Ju_n) + \xi_3 \Phi(Ku_n, Ju_n)$

Taking limit as $n \to \infty$, we get
$\Phi(\beta, \beta) \leq (\xi_1 + \xi_2 + \xi_3)\, \Phi(\beta, \beta) = 0$

Since, from the definition of weakly commutative mappings, we have
$\Phi(K(J(u_n)), J(K(u_n))) \leq \Phi(K(u_n), J(u_n))$
$\Phi(K(\beta), J(\beta)) \leq \Phi(\beta, \beta)$
$K(\beta) = J(\beta)$

Clearly, β is a coincidence point of mappings J and K. So, $K(J\beta) = J(K\beta) = J(J\beta)$.

Therefore, we get
$\Phi(J\beta, J\beta) \leq \xi_1 \Phi(K\beta, K\beta) + \xi_2 \Phi(K\beta, J\beta) + \xi_3 \Phi(K\beta, J\beta)$
$\Phi(J\beta, J\beta) \leq (\xi_1 + \xi_2 + \xi_3)\, \Phi(J\beta, J\beta)$
$(1 - \xi_1 - \xi_2 - \xi_3) \Phi(J\beta, J\beta) \leq 0 \in \Phi(J\beta, J\beta) = 0$

Similarly, we can show that
$\Phi(J(J\beta), J(J\beta)) = 0$

Hence,
$\Phi(J\beta, J(J\beta)) \leq \xi_1 \Phi(K\beta, K(J\beta)) + \xi_2 \Phi(K\beta, J\beta) + \xi_3 \Phi(K(J\beta), J(J\beta))$
$\leq \xi_1 \Phi(K\beta, K(J\beta)) + \xi_2 \Phi(K\beta, J\beta) + \xi_3 \Phi(K(J\beta), J(J\beta))$
$\leq \xi_1 \Phi(J\beta, J(J\beta))$

This implies,

$\Phi(J\beta, J(J\beta))(1 - \xi_1) \leq 0 \leq J\beta = J(J\beta) = K(J\beta)$

i.e., $J(\beta)$ is a common fixed point of mappings J and K.

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Now, to show the uniqueness of common fixed point, suppose that $P_1 = K(P_1) = J(P_1)$ and $P_2 = K(P_2) = J(P_2)$. Then from (3) we have

$$\Phi(P_1, P_2) = \Phi(JP_1, JP_2)$$
$$\leq \xi_1 \Phi(KP_1, KP_2) + \xi_2 \Phi(KP_1, JP_1) + \xi_3 \Phi(KP_2, JP_2)$$
$$\leq (\xi_1 + \xi_2 + \xi_3)\, \Phi(P_1, P_2)$$
$$\leq \alpha \Phi(P_1, P_2)$$

Where, $\alpha = (\xi_1 + \xi_2 + \xi_3) < 1$. So we get

$\Phi(P_1, P_2)\,(1-\alpha) \leq 0$, implies that $P_1 = P_2$ since $\alpha < 1$.

Hence, mappings J and K have a unique common fixed point.

4. Conclusion

This paper is aimed at introducing the perception of weakly commutative mappings in digital metric space and by using these mappings and their variants, establish a digital common fixed point theorem in this space. Our result broaden and extend many prevailing known results in the literature. This result is an application in fixed point theory. Which can be used to compress digital images and can be beneficial in processing and redefining image storage.

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