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Transformation of Parabolic Partial Differential Equations into Heat Equation Using Hopf Cole Transform

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Abstract: ***This paper is concerned with the transformation of four parabolic partial differential equations to heat equations using Hopf Cole transform. The transformation shows that Hopf Cole transform is a good mathematical transformation tool that can be applied to any parabolic partial differential equation that needs to be transformed to heat equation to obtain the solution.***

Keywords: Hopf Cole transform, Parabolic partial differential equations, Black–Scholes equation, Variable coefficient partial differential equation, Constant coefficient partial differential equation.

1. Introduction

Transformation methods provide a work way between the known method, separation of variables method and numerical techniques for solving linear and non linear partial differential equations. Different transform methods can be effective for a wider class of problems. Hopf [2] introduced the transform in 1950 where he transformed the nonlinear Bergers' equation to a heat equation. Cole [4] studied a quasi-linear parabolic equation using the transform. Tai-Pinj Liu [9] transformed Bergers equation to linear heat equation and solved. Researchers in mathematical finance transform Black Scholes differential equation into heat equation using exponential transforms, Mellin transform etc. Fares *et al* [3] transformed Black Scholes to heat equation with Mellin transform. Yesiltas [5] in his investigation, transformed Black Scholes equation to heat equation with Darboux transformation. Sozhaeswari *et al* [6] in their study on the impact of Nonlinear Source term in Black-scoholes applied an exponential transform to transform the nonlinear Black Scholes to heat equation. Akeju [1] in his study, transformed Black Scholes Model into heat equation by the application of logarithmic transform and other series of transformations. Kumar and Yildirim [8] transformed Black Scholes European Option Pricing equation to heat equation with Laplace transform in their research. Frontezak [7] used Mellin transform to transform the Black Scholes equation and solved thereafter.

2. Definition of Hopf Cole Transformation

Hopf (1950) and Cole (1951) defined Hopf Cole transformation as

$$v(x,t) = -c\frac{\partial \ln \phi}{\partial x} \tag{1}$$

where $\phi = \phi(x,t)$

Hopf Cole transform, equation (1) is be applied to constant and variable coefficients parabolic partial differential equations to transform them to heat equations which are easy to solve. In this research, the Hopf Cole transformation will be taken as

$$v(x,t) = \frac{\partial \ln \phi}{\partial x} \tag{2}$$

where $c = -1$

3. Constant coefficients parabolic partial differential equations

(a) $$\frac{\partial u}{\partial t} = k\left\{\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu\right\} \tag{3}$$

From equation (2),

$$u(x,t) = \frac{\partial \ln \phi}{\partial x} \tag{4}$$

Applying equation (4) on equation (3), we have

$$\phi_{xt}\phi^{-1} - \phi^{-2}\phi_t\phi_s - k\left\{\phi_{xxx}\phi^{-1} - \phi^{-2}\phi_x\phi_{xx} - 2\phi_x\phi_{xx}\phi^{-2} + 2\phi^{-3}(\phi_x)^2\right\} - \alpha\left\{\phi_{xx}\phi^{-1} - \phi^{-2}(\phi_x)^2\right\} - \beta\phi_x\phi^{-1} = 0 \tag{5}$$

where $\alpha = ka,\ \beta = kb$

Further simplification of equation (5) gives

$$\phi_{xt}\phi^2 - \phi\phi_t\phi_s - k\left\{\phi_{xxx}\phi^2 - 3\phi_x\phi_{xx}\phi + 2(\phi_x)^2\right\} - \alpha\left\{\phi_{xx}\phi^2 - \phi(\phi_x)^2\right\} - \beta\phi_x\phi^2 = 0 \tag{6}$$

Equation (5) can be written as

$$\phi^2(\phi_{xt} - k\phi_{xxx} - \alpha\phi_{xx} - \beta\phi_x) - \phi\phi_x(\phi_t - 3k\phi_{xx}) - (\phi_x)^2(2k + \alpha\phi) = 0 \tag{7}$$

Equate each term on the left hand side of equation (7) to zero

$$\phi_{xt} - k\phi_{xxx} - \alpha\phi_{xx} - \beta\phi_x = 0 \tag{8a}$$

$$\phi_t - 3k\phi_{xx} = 0 \tag{8b}$$

$$2k + \alpha\phi = 0 \tag{8c}$$

Equation (8a) is still of the form of the given equation under consideration. Equation (8c) is not a differential equation and solution of it might not be the solution of equation (3).

Equation (8b) is of the form of heat equation and its' solution is the solution of equation (3).

Equation (8b) can be written as

$$\frac{\partial u}{\partial t}+c\frac{\partial^2 u}{\partial x^2}=0 \quad (9)$$

where $c=-3k$

(b) $$\frac{\partial u}{\partial t}-4\frac{\partial^2 u}{\partial x^2}-2\frac{\partial u}{\partial x}-u=0 \quad (10)$$

Applying equation (3) on equation (10), equation (10) becomes

$$\phi_{xt}\phi^{-1}-\phi^{-2}\phi\phi_x-4\left\{\phi_{xxx}\phi^{-1}-3\phi_x\phi_{xx}\phi^{-2}+2\phi^{-3}(\phi_x)^2\right\}$$
$$-2\left\{\phi_{xx}\phi^{-1}-\phi^{-2}(\phi_x)^2\right\}-\phi_x\phi^{-1}=0 \quad (11)$$

Further simplification of equation (11) gives

$$\phi^2(\phi_{xt}-4\phi_{xxx}-2\phi_{xx}-\phi_x)-\phi\phi_x(\phi_t+12\phi_{xx})-$$
$$(\phi_x)^2(8+\phi)=0 \quad (12)$$

Equate each term on the left hand side of equation (12) to zero, we have

$$\phi_{xt}-4\phi_{xxx}-2\phi_{xx}-\phi_x=0 \quad (13a)$$
$$\phi_t+12\phi_{xx}=0 \quad (13b)$$
$$8+\phi=0 \quad (13c)$$

Equation (13b) is of the form of heat equation and can be written as

$$\frac{\partial u}{\partial t}+12\frac{\partial^2 u}{\partial x^2}=0 \quad (14)$$

4. Variable coefficients parabolic partial differential equations

(c) $$\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 v}{\partial s^2}+rs\frac{\partial v}{\partial s}-rv=0 \quad (15)$$

Equation (15) is Black Scholes equation.

Transforming equation (15) to heat equation, using equation (2), we let

$$v(s,t)=\frac{\partial \ln\phi}{\partial s} \quad (16)$$

where $\phi=\phi(s,t)$

Substitute v, v_t, v_s, v_{ss} in equation (15) using equation (16) and multiply by ϕ^3

$$\phi_{st}\phi^2-\phi\phi_t\phi_s+\frac{1}{2}\sigma^2 s^2\left\{\phi_{sss}\phi^2-3\phi\phi_s\phi_{ss}+2(\phi_s)^2\right\}+$$
$$rs\left\{\phi_{ss}\phi^2-\phi(\phi_s)^2\right\}-r\phi_s\phi^2=0 \quad (17)$$

Equation (17) can be rewritten as

$$\phi^2\left(\phi_{st}+\frac{1}{2}\sigma^2 s^2\phi_{sss}+rs\phi_{ss}-r\phi_s\right)-\phi_s\phi(\phi_t+$$
$$3\sigma^2 s^2\phi_{ss})+(\phi_s)^2(\sigma^2 s^2-rs\phi)=0 \quad (18)$$

Equate each term on the left hand side to zero

$$\phi_{st}+\frac{1}{2}\sigma^2 s^2\phi_{sss}+rs\phi_{ss}-r\phi_s=0 \quad (19a)$$
$$\phi_t+3\sigma^2 s^2\phi_{ss}=0 \quad (19b)$$
$$\sigma^2 s^2-rs\phi=0 \quad (19c)$$

Equation (19b) is a heat equation of the form

$$\frac{\partial \phi}{\partial t}+k^2\frac{\partial^2\phi}{\partial s^2}=0 \quad (20)$$

where $k^2=\frac{3}{2}\sigma^2 s^2$

(d) $$\frac{\partial u}{\partial t}+2\xi\varphi x\frac{\partial^2 u}{\partial x^2}+\{\xi\varphi-(\xi+\eta)x\}\frac{\partial u}{\partial x}-axu=0 \quad (21)$$

Equation (21) can be a model of energy equation to study electricity prices

Applying equation (2) on equation (21), equation (21) becomes

$$\phi_{xt}\phi^{-1}-\phi^{-2}\phi_t\phi_x+2\xi\varphi x\left\{\phi_{xxx}\phi^{-1}-3\phi_x\phi_{xx}\phi^{-2}+2\phi^{-3}(\phi_x)^3\right\}+$$
$$\{\xi\varphi-(\xi+\eta)x\}\left\{\phi_{xx}\phi^{-1}-\phi^{-2}(\phi_x)^2\right\}-ax\phi_x\phi^{-1}=0 \quad (22)$$

Further simplification of equation (22) gives

$$\phi_{xt}\phi^2-\phi\phi_t\phi_x+2\xi\varphi x\left\{\phi_{xxx}\phi^2-3\phi_x\phi_{xx}\phi+2(\phi_x)^3\right\}+$$
$$\{\xi\varphi-(\xi+\eta)x\}\left\{\phi_{xx}\phi^2-\phi(\phi_x)^2\right\}-ax\phi_x\phi^2=0 \quad (23)$$

Equation (23) can be written as

$$\phi^2\left[\phi_{xt}+2\xi\varphi x\phi_{xxx}+\{\xi\varphi-(\xi+\eta)x\}\phi_{xx}-ax\phi_x\right]$$
$$-\phi\phi_x(\phi_t+6\xi\varphi x\phi_{xx})+$$
$$(\phi_x)^2\left[4\xi\varphi x-\{\xi\varphi-(\xi+\eta)x\}\phi\right]=0 \quad (24)$$

Equate each term on the left hand side of equation (24) to zero

$$\phi_{xt}+2\xi\varphi x\phi_{xxx}+\{\xi\varphi-(\xi+\eta)x\}\phi_{xx}-ax\phi_x=0 \quad (25a)$$
$$\phi_t+6\xi\varphi x\phi_{xx}=0 \quad (25b)$$
$$4\xi\varphi x-\{\xi\varphi-(\xi+\eta)x\}\phi=0 \quad (25c)$$

Equation (25b) is a heat equation and can be written as

$$\frac{\partial u}{\partial t}+k\frac{\partial^2 u}{\partial x^2}=0 \quad (26)$$

where $k=6\xi\varphi x$.

It is well known that heat equation

$$\frac{\partial y}{\partial t}+\lambda\frac{\partial^2 y}{\partial x^2}=0 \quad (27a)$$

With initial condition, say

$$y(x,0)=g(x) \quad (27b)$$

Has unique solution

$$y(x,t)=\int_{-\infty}^{\infty} f(\eta)g(x-\eta,t)d\eta \quad (28)$$

where $g(x-\eta,t)$ is the heat kernel.

5. Conclusion

Transformation of linear constant and variable constant coefficients parabolic partial differential equations have been transformed to one dimensional heat equation using Hopf Cole transformation. The transformations show that Hopf Cole transformation can be applied to other linear parabolic partial differential equations including Black Scholes and not only non linear partial differential equation such as Burgers equation.

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