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On Property (Baw1)

Neeru Kashyap

Department of Mathematics, Bhaskaracharya College of Applied Sciences, University of Delhi, Dwarka, New Delhi-110075, India
Email: *neeru.sharma[a]bcas.du.ac.in*

Abstract: ***This paper introduces the notion of property (Baw1), which is an extension of the property (Baw) defined and studied in [14]. We establish for a bounded linear operator defined on a Banach space the necessary and sufficient conditions for which the property (Baw1) holds. We discuss the property (Baw1) for operators satisfying the single valued extension property (SVEP).Certain conditions are explored on Hilbert space operators T and S so that $T \oplus S$ obeys the property (Baw1).We also study the preservation of the property (Baw) under perturbations by finite rank and nilpotent operators.***

Keywords: Weyl's theorem; Generalized Weyl's theorem; Generalized Browder's theorem; SVEP; Property (Baw1); Property (Baw); Finitely polaroid operators.

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1. Introduction and Preliminaries

Let $B(X)$ denote the Banach algebra of all bounded linear operators on an infinite-dimensional complex Banach space X. For an operator $T \in B(X)$, let $T^*, N(T)$, $R(T), \sigma(T)$ *and* $\sigma_a(T)$denote respectively the adjoint, the null space, the range space, the spectrum and the approximate spectrum of T. Let $\propto(T)$ *and* $\beta(T)$be the nullity and deficiency of T defined by$\propto(T) = dimN(T)$and $\beta(T)$ =codim $R(T)$. If the range$R(T)$ of T is closed and $\propto(T) < \infty$(resp. $\beta(T) < \infty$), *then* T is said to be an upper (resp., a lower) semi-Fredholm operator. Let $USF(X)$ denote the class of all upper semi-Fredholm operators.An operator $T \in B(X)$ is said to be semi-Fredholm if T is either an upper or a lower semi- Fredholm and the index of T is defined by ind(T) =$\propto(T) - \beta(T)$.

If $T \in B(X)$ is both upper and lower semi-Fredholm then T is said to be the Fredholm operator. An operator $T \in B(X)$ is called a Weyl operator if it is a Fredholm operator of index zero. The Weyl Spectrum of T is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$.

Denote by $USF^-(X)$ the class of all upper semi B-Fredholm operators with an index less than or equal to 0. Set $\sigma_{usf^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin USF^-(X)\}$.

Following Coburn [9],we say that Weyl's theorem holds for $T \in B(X)$ if $\sigma(T) \setminus \sigma_W(T) = E_0(T)$, where $E_0(T) = \{\lambda \in iso\sigma(T): 0 < \alpha(T - \lambda I) < \infty\}$.Here and elsewhere for $A \subset \mathbb{C}$, isoA denotes the set of all isolated points of A and accA denotes the set of all points of accumulation of A.According toRakočević [17] an operator $T \in B(X)$is said to satisfy a-Weyl's theorem if,$\sigma_a(T) \setminus \sigma_{usf^-}(T) = E_0^a(T)$, where $E_0^a(T) = \{\lambda \in iso\ \sigma_a(T): 0 < \alpha(T - \lambda I) < \infty\}$.

For a bounded linear operator $T \in B(X)$ and a non-negative integer n, we define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into itself (in particular $(T_0 = T)$. If for some integer n, the range space $R(T^n)$ is closed and T_n is an upper (resp., a lower) semi-Fredholm operator, then T is called an upper (resp., a lower) semi B - Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. From [8, Proposition 2.1] if T_n is a semi- Fredholm operator then T_m is also a semi – Fredholm operator for each $m \geq n$ and $ind(T_m) = ind(T_n)$. Thus, the index of a semi-B-Fredholm operator T is defined as the index of the semi- Fredholm operator T_n. (see [7,8]). An operator $T \in B(X)$ is called a B-weyl operator if it is a B- Fredholm operator of index 0. The B- Weyl spectrum $\sigma_{BW}(T)$ of T is defined as $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not } B - Weyl \text{ operator}\}$. Let $USBF^-(X)$ be the class of all upper semi-B-Fredholm operators with an index less than or equal to 0. The upper B-Weyl spectrum of T is defined by $\sigma_{usbf^-}(T) = \{\lambda \in \mathbb{C}: T - \lambda I \notin USBF^-(X)\}$.

Let $p(T)$: = asc(T) be the ascent of an operator T i.e., the smallest nonnegative integer n such that $N(T^n) = N(T^{n+1})$. If such an integer does not exist we put asc(T)= ∞. Analogously, let $q(T) :=$ dsc(T) be the descent of an operator$T$i.e. the smallest non-negative integer such that $R(T^n) = R(T^{n+1})$and if such an integer does not exist we put dsc(T) = ∞. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$.An operator T is called Drazin invertible if it has finite ascent and descent. The Drazin spectrum of T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}$. We observe $\sigma DT=\sigma(T)$ \$\pi(T)$,where $\pi(T)$is the set of poles of T.

An operator $T \in B(X)$ is called an upper semi-Browder if it is an upper semi-Fredholm of finite ascent, and is called Browder if it is a Fredholm of finite ascent and descent. The Browder spectrum of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : \text{T} - \lambda\text{I is not Browder}\}$.Define the set *LD(X)*as follows:

$$LD(X) = \{T \in B(X) : \propto(T) < \infty \text{ and is } R(T^{\propto(T)+1} \text{ closed}\}$$

and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(X)\}$.An operator $T \in B(X)$ is said to be left Drazin invertible if $T \in LD(X)$. We say that$\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I \in LD(X)$and that$\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$. Let $\pi^a(T)$ denote the set of all left poles of T and $\pi_0^a(T)$ denotes the set of all left poles of T of finite rank.Following [7] , we say that generalized a-Browder's theorem holds for T if $\sigma_a(T) \setminus$

$\sigma_{usbf^-}(T) = \pi^a(T)$ and that a-Browder's theorem holds for T if$\sigma_a(T) \setminus \sigma_{usf^-}(T) = \pi_0^a(T)$. It is proved in [3,Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem.

Given $T \in B(X)$, we say that generalized Browder's theorem holds for T if $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$, and that Browder's theorem holds for T if $\sigma(T) \setminus \sigma_W(T) = \pi_0(T)$, where $\pi_0(T)$is the set of all poles of T of finite rank. It is proved in [3, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem.

We say that T obeys generalized a-Weyl theorem if $\sigma_a(T) \setminus \sigma_{usbf^-}(T) = E^a(T)$, Where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$and that generalized Weyl's theorem holds for T if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, Where $E(T)$ is the set of isolated eigenvaluesof T [7, Definition2.13]. Generalized a-Weyl's theorem has been studied in [3]. In [7, Theorem 3.11], it is shown that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem.Generalized Weyl's theorem has been studied in [2,4-8] and the references therein. Berkani and Koliha [7] proved that generalized Weyl's theorem ⇒ Weyl's theorem.

The single valued extension property was introduced by Dunford ([11],[12]) and it plays an important role in local spectral theory and Fredholm theory ([1],[15]).

The operator $T \in B(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$(abbreviated SVEP at $\lambda_0 \in \mathbb{C}$)if for every open disc U of λ_0 the only analytic function $f: U \to X$ which satisfies the equation $(T - \lambda I)f(\lambda)=0$ for all $\lambda \in U$, is the function$f \equiv 0$.

An operator $T \in B(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. An operator $T \in B(X)$ has SVEP at every point of the resolvent $\rho(T)= \mathbb{C}\setminus\sigma(T)$. Every operator Thas SVEP at an isolated point of the spectrum. Duggal [10] gave the following important result:

Theorem1.1 ([10, Proposition 3.10]). The following statements are equivalent.
(i) T satisfies generalized a-Browder's theorem
(ii) T has SVEP at points $\lambda \notin \sigma_{usbf^-}(T)$.

2. Property (Baw1)

Property (Baw) has been defined in [14] as

Definition 2.1([14, Definition 2.1]).A bounded linear operator $T \in B(X)$is said to satisfy property (Baw) if $\sigma_a(T) \setminus \sigma_{usbf^-}(T) = E_0^a(T)$.

We now a define property (Baw1) for a bounded linear operators T as an extension of generalized Weyl's theorem. We establish the necessary and sufficient conditions for which this propertyholds. We prove that T satisfies property (Baw1) if and only if generalized a-Browder's theorem holds for T and $\pi^a(T) \subseteq E_0^a(T)$.

Definition2.2. A bounded linear operator $T \in B(X)$is said to satisfy property (Baw1) if $\sigma_a(T) \setminus \sigma_{usbf^-}(T) \subseteq E_0^a(T)$.

The following example shows that property (Baw1) does not imply property (Baw) in general.

Example 2.1. Let $R \in B(l^2(N))$ be the right shift and let L be the weighted unilateral shift defined by
$L(x_1, x_2, x_3 \ldots\ldots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \ldots)$ for all $(x_1, x_2, \ldots\ldots) \in l^2$ (N).

Consider the operator$T$ defined on $l^2(N) \oplus l^2(N)$by by$T = R \oplus L, \sigma(T) = D(0,1)$ is the closed unit disc in $\mathbb{C}$. On the other hand$\sigma_a(T) = \sigma_{usbf^-}(T) = C(0,1) \cup \{0\}$. However,$E_0^a(T) = \{0\}$. Thus T satisfies property (Baw1) but the property(Baw) is not satisfied.

Theorem 2.1.*Property (Baw) holds for T if and only if T satisfies property (Baw1) and*$\sigma_{usbf^-}(T) \cap E_0^a(T) = \emptyset$.

Proof: Suppose that T satisfies property (Baw), then property(Baw1) holds for T and$\sigma_{usbf^-}(T) \cap E_0^a(T) = \emptyset$.For the converse, if $\lambda \in E_0^a(T)$, $\lambda \notin \sigma_{usbf^-}(T)$since$\sigma_{usbf^-}(T) \cap E_0^a(T) = \emptyset$. Thus $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T)$. Hence $E_0^a(T) \subseteq \sigma_a(T) \setminus \sigma_{usbf^-}(T)$.

Theorem2.2.*If $T \in B(X)$ satisfies property (Baw1). Thengeneralized a-Browder's theorem holds for $T$and* $\sigma_a(T) = \sigma_{usbf^-}(T) \cup E_0^a(T)$.

Proof.By Theorem 1.1, it is sufficient to prove that T has SVEP at every $\lambda \notin \sigma_{usbf^-}(T)$. Let us assume that $\lambda \notin \sigma_{usbf^-}(T)$.
Case (i): If $\lambda \notin \sigma_a(T)$then T has SVEP at λ.
Case (ii): If $\lambda \in \sigma_a(T)$and suppose that T satisfies property (Baw1) then

$$\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T) \subseteq E_0^a(T).$$

Hence, $\lambda \in \sigma_a^{iso}(T)$, so, also, in this case, T has SVEP at λ. To prove$\sigma_a(T) = \sigma_{usbf^-}(T) \cup E_0^a(T)$, we observe that$\sigma_{usbf^-}(T) \cup E_0^a(T) \subseteq \sigma_a(T)$for every $T \in B(X)$. For the reverse inclusion, consider$\lambda \in \sigma_a(T)$. If $\lambda \notin \sigma_{usbf^-}(T)$then $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T)$. As T satisfies property (Baw1), therefore $\lambda \in E_0^a(T)$. Thus $\sigma_a(T) = \sigma_{usbf^-}(T) \cup E_0^a(T)$.

Now we give a characterization of property (Baw1):

Theorem2.3.*If$T \in B(X)$, then the following statements are equivalent:*
(i) T satisfies property (Baw1),
(ii) generalized a-Browder's theorem holds for $T$and$\pi^a(T) \subseteq E_0^a(T)$.

Proof. (i)⇒ (ii) Assume that T satisfies property (Baw1). By Theorem (2.2) it is sufficient to prove that $\pi^a(T) \subseteq E_0^a(T)$.Let $\lambda \in \pi^a(T)$=$\sigma_a(T)\setminus\sigma_{usbf^-}(T) \subseteq E_0^a(T)$. (ii) ⇒ (i). If $\lambda \in \sigma_a(T)\setminus\sigma_{usbf^-}(T)$.Then generalized a-Browder's theorem implies that $\lambda \in \pi^a(T) \subseteq E_0^a(T)$. Thus $\sigma_a(T)\setminus \sigma_{usbf^-}(T) \subseteq E_0^a(T)$.

Theorem 2.4: *Let $T \in B(X)$. If T has SVEPat points in $\sigma_a(T)\backslash\sigma_{usbf^-}(T)$, then T Satisfies property (Baw1)if and only if $\pi^a(T) \subseteq E_0^a(T)$.*

Proof: The hypothesis that T has SVEP at $\sigma_a(T)\backslash \sigma_{usbf^-}(T)$ implies that T satisfies generalized a- Browder's theorem (see Theorem 1.1)

Hence if $\pi^a(T) \subseteq E_0^a(T)$ then $\sigma_a(T)\backslash\sigma_{usbf^-}(T) = \pi^a(T) \subseteq E_0^a(T)$.

Definition 2.3: Operators$S, T \in B(X)$are said to be injectively interwined, denoted,$S \prec_i T$, if there exists an injection$U \in B(X)$such that $TU = US$.

If $S \prec_i T$, then T has SVEP at a point λ implies S has SVEP at λ.

To see this, let T have SVEP at λ and let U be an open neighbourhood of λ and let $f: U \to X$ be an analytic function such that $(S-\mu)f(\mu) = 0$ for every $\mu \in U$.Then $U(S-\mu)f(\mu) = (T-\mu)Uf(\mu) = 0 \Rightarrow Uf(\mu) = 0$. Since U is injective,$f(\mu) = 0$, i.e.,S has SVEP at λ.

Theorem 2.5: *Let $S, T \in B(X)$.If T has SVEP and $S \prec_i T$,then property (Baw1) holds for S if and only if $\pi^a(S) \subseteq E_0^a(S)$.*

Proof. Suppose that T has SVEP.Since$S \prec_i T$,therefore$S$ has SVEP.Hence the result follows from Theorem2.4.

Definition 2.4: An operator $T \in B(X)$ is said to be left polaroid if all the isolated points of its approximate spectrum are left poles$\sigma_a^{iso}(T) \subseteq \pi^a(T)$.

Theorem 2.6: *Let $T \in B(X)$be left polaroid and satisfy property (Baw1), then generalized a-Weyl's theorem holds for T.*

Proof: T is a polaroid and satisfies property (Baw1) if and only if

$$\sigma_a(T)\backslash\sigma_{usbf^-}(T) \subseteq E_0^a(T) \subseteq E^a(T) = \pi^a(T) = \sigma_a(T)\backslash\sigma_{usbf^-}(T)$$

(Since$T$ satisfies generalized a-Browder's theorem by Theorem 2.3.

3. Property (Baw1) for direct sums

Let H and K be infinite-dimensional Hilbert spaces. In this section, we show that if T and S are two operators on H and K respectively and at least one of them satisfies property (Baw1), then their direct sum $T \oplus S$ obeys property (Baw1). We have also explored various conditions on T and S so that $T \oplus S$ satisfies the property (Baw1).

Theorem 3.1.*Suppose that property (Baw1) holds for $T \in B(H)$ and $S \in B(K)$. If T and S are a-isoloid and$\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)$, then property (Baw1) holds for $T \oplus S$.*

Proof. We know $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pair of operators.

If T and S are a-isoloid, then

$$E_0^a(T \oplus S) = [E_0^a(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap E_0^a(S)] \cup [E_0^a(T) \cap E_0^a(S)]$$

where $\rho_a(.) = \mathbb{C}\backslash\sigma_a(.)$.

If property(Baw1)holds for$T$ and S, then

$$[\sigma_a(T) \cup \sigma_a(S)]\backslash[\sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)] \subseteq [E_0^a(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap E_0^a(S)] \cup [E_0^a(T) \cap E_0^a(S)].$$

Thus $\sigma_a(T \oplus S)\backslash\sigma_{usbf^-}(T \oplus S) \subseteq [E_0^a(T \oplus S)$.Hence property(Baw1) holds for $T \oplus S$.□

Theorem 3.2.*Suppose $T \in B(H)$has no isolated point in its spectrum and $S \in B(K)$ satisfies property (Baw1). If$\sigma_{usbf^-}(T \oplus S) = \sigma_a(T) \cup \sigma_{usbf^-}(S)$, then property (Baw1) holds for $T \oplus S$.*

Proof. As $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pair of operators, we have

$$\sigma_a(T \oplus S)\backslash\sigma_{usbf^-}(T \oplus S) = [\sigma_a(T) \cup \sigma_a(S)]\backslash[\sigma_a(T) \cup \sigma_{usbf^-}(S)]$$
$$=\sigma_a(S)\backslash[\sigma_a(T) \cup \sigma_{usbf^-}(S)]$$
$$=[\sigma_a(S)\backslash\sigma_{usbf^-}(S)]\backslash\sigma_a(T)$$
$$\subseteq E_0^a(S) \cap \rho_a(T)$$

where$\rho_a(T) = \mathbb{C}\backslash\sigma_a(T)$.

Now$\sigma_a^{iso}(T)$ be the set of isolated points of$\sigma_a(T)$ and $\sigma_a^{iso}(T \oplus S)$is the set of isolated points of$\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$. If $\sigma_{iso}^a(T) = \phi$ it implies that $\sigma(T) = \sigma_{acc}(T)$, where $\sigma_{acc}(T) = \sigma_a(T)\backslash\sigma_a^{iso}(T)$ is the set of all accumulation points of $\sigma_a(T)$.Thus we have

$$\sigma_a^{iso}(T \oplus S) = [\sigma_a^{iso}(T) \cup \sigma_a^{iso}(S)]\backslash[(\sigma_a^{iso}(T) \cap \sigma_{acc}(S)) \cup (\sigma_{acc}(T) \cap \sigma_a^{iso}(S))]$$
$$=(\sigma_a^{iso}(T)\backslash\sigma_{acc}(S)) \cup (\sigma_a^{iso}(S)\backslash\sigma_{acc}(T))$$
$$=\sigma_a^{iso}(S)\backslash\sigma_a(T)$$
$$=\sigma_a^{iso}(S) \cap \rho_a(T).$$

Let $\sigma_P(T)$ denote the point spectrum of T and $\sigma_{PF}(T)$ denote the set of all eigen values of T of finite multiplicity.

We have that $\sigma_P(T \oplus S) = \sigma_P(T) \cup \sigma_P(S)$ and $dim\, N(T \oplus S) = dim\, N(T) + dim\, N(S)$for every pair of operators, so that

$$\sigma_{PF}(T \oplus S) = \{\lambda \in \sigma_{PF}(T) \cup \sigma_{PF}(S): dimN(\lambda I - T) + dim\, N(\lambda I - S) < \infty\}$$

Therefore,

$$E_0^a(T \oplus S) = \sigma_a^{iso}(T \oplus S) \cap \sigma_{PF}(T \oplus S)$$
$$=\sigma_a^{iso}(S) \cap \rho_a(T) \cap \sigma_{PF}(S)$$
$$= E_0^a(S) \cap \rho_a(T).$$

Thus,$\sigma_a(T \oplus S)\backslash\sigma_{usbf^-}(T \oplus S) \subseteq E_0^a(T \oplus S)$. Hence, $T \oplus S$ satisfies the property (Baw1).

Let $\sigma_1(T)$ denote the compliment of $\sigma_{usbf^-}(T)$ in $\sigma_a(T)$i.e.$\sigma_1(T) = \sigma_a(T)\backslash\sigma_{usbf^-}(T)$. A straight forward application of Theorem3.2leads to the following corollaries.

Corollary 3.1: *Suppose* $T \in B(H)$ *is such that* $\sigma_a^{iso}(T) = \phi$ *and* $S \in B(K)$ *satisfies property (Baw1) with* $\sigma_a^{iso}(S) \cap \sigma_{PF}(S) = \phi$ *and* $\sigma_1(T \oplus S) = \varphi$, *then* $T \oplus S$ *satisfies property(Baw1).*

Proof: Since Ssatisfies property (Baw1), therefore given condition $\sigma_a^{iso}(S) \cap \sigma_{PF}(S) = \phi$implies that$\sigma_a(S) = \sigma_{usbf^-}(S)$. Now $\sigma_1(T \oplus S) = \phi$ gives that $\sigma_a(T \oplus S) = \sigma_{usbf^-}(T \oplus S) = \sigma_a(T) \cup \sigma_{usbf^-}(S)$. Thus, from Theorem 3.2 we have that $T \oplus S$ satisfies property(Baw1).

Corollary 3.2: *Suppose* $T \in B(H)$ *is such that* $\sigma_1(T) \cup \sigma_a^{iso}(T) = \phi$ *and* $S \in B(K)$ *satisfies property (Baw1). If*$\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)$, *then property (Baw1) holds for* $T \oplus S$.

Theorem 3.3: *Suppose* $T \in B(H)$ *is an isoloid operator that satisfies property (Baw1), then* $T \oplus S$ *satisfies property (Baw1) whenever* $S \in B(K)$ *is a normal operator and satisfies property (Baw1).*

Proof: If $S \in B(K)$ is normal, then S (also,S^*) has SVEP, and $ind(S - \lambda) = 0$ for every λ such that $S - \lambda$ is B-Fredholm. Observe that $\lambda \notin \sigma_{usbf^-}(T \oplus S) \Leftrightarrow T - \lambda$ and $S - \lambda$ are B- Fredholm and ind$(T - \lambda) + indS - \lambda) = ind(T - \lambda) = 0$.
$\Leftrightarrow \lambda \notin$
$\{\sigma_a(T) \backslash \sigma_{usbf^-}(T)\} \cap$
$\{\sigma_a(S) \backslash \sigma_{usbf^-}(S)\}$. Hence,$\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)$.

It is well known that the isolated points of the spectrum of a normal operator are simple poles of the resolvent of the operator(implies S is a-isoloid). Hence the result follows from Theorem 3.1.

4. Property (Baw) and perturbations

In this section, we study the preservation of property (Baw) under perturbations by finite rank and nilpotent operators.

Theorem4.1: *Let* $T \in B(X)$. *If T has property (Baw) and F is a finite rank operator in* $B(X)$ *that commutes with T, then T+F has property (Baw) if and only if* $\pi^a(T + F) = E_0^a(T + F)$.

Proof: If T+F has property (Baw), then $\pi^a(T + F) = E_0^a(T + F)$. Conversely, if $\pi^a(T + F) = E_0^a(T + F)$. Since F is a finite rank operator in B(X) that commutes with T, therefore $\sigma_{usbf^-}(T) = \sigma_{usbf^-}(T + F)$ and $\sigma_{LD}(T) = \sigma_{LD}(T + F)$[7, Theorem 4.3].As T satisfies generalized a-Browder's theorem, therefore $\sigma_{usbf^-}(T) = \sigma_{LD}(T)$. Now $\sigma_a(T + F) \backslash \sigma_{usbf^-}(T + F)$=$\sigma_a(T + F) \backslash \sigma_{LD}(T + F) = \pi^a(T + F) = E_0^a(T + F)$. Therefore, T+F the satisfies property (Baw1).

Theorem 4.2: *Let* $T \in B(X)$ *and let* N *be a nilpotent operator commuting with T. If T satisfies property (Baw), then the following statements are equivalent.*

(i) *T+N satisfies property (Baw).*
(ii) $\sigma_{usbf^-}(T + N) = \sigma_{usbf^-}(T)$.
(iii) $E_0^a(T) = \pi^a(T + N)$.

Proof: (i) ↔(ii) Assume that $T + N$ satisfies property (Baw), then$\sigma_a(T + N) \backslash \sigma_{usbf^-}(T + N) = E_0^a(T + N)$. As $\sigma_a(T + N) = \sigma a T$ and $E0aT+N = E0aT$.Then, $\sigma a(T) \backslash \sigma_{usbf^-}(T + N) = E_0^a(T)$. Since T satisfies property (Baw), then$\sigma_a(T) \backslash \sigma_{usbf^-}(T) = E_0^a(T)$. So $\sigma_{usbf^-}(T + N) = \sigma_{usbf^-}(T)$.Conversely assume that $\sigma_{usbf^-}(T + N) = \sigma_{usbf^-}(T)$, then as T satisfies property (Baw)it follows that T+N also satisfies property (Baw).

(i)↔(iii) Assume that $T + N$ satisfies property (Baw), then $\pi^a(T + N) = E_0^a(T + N)$ [14, Theorem 2.5]. Thus $E_0^a(T) = \pi^a(T + N)$. Conversely assume that $E_0^a(T) = \pi^a(T + N)$. If T satisfies property (Baw), then T satisfies the generalized a- Browder theorem [14, Theorem 2.5]. As T+N satisfies property(Baw), T+N satisfies generalized a-Browder's theorem that is $\sigma_a(T + N) \backslash \sigma_{usbf^-}(T + N) = \pi^a(T + N)$. By assumption $E_0^a(T) = \pi^a(T + N)$, it follows that $\sigma_a(T + N) \backslash \sigma_{usbf^-}(T + N) = E_0^a(T + N)$ and so T+N satisfies property (Baw).

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