Cycling: An Appraisal of Convergence Patterns with Respect to Even and Odd *N*-Point Designs

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Abstract: ***Cycling, a problem associated with the construction of D-optimal designs, slows down the convergence of a sequence to a desired optimum, whenever it occurs in a variance exchange process. In this paper, we study the pattern of convergence induced by cycling especially when the size, N, of the design measure, ξ_N, is either even or odd. The linear, interactive and quadratic portions of a two-factor response function, are used in the investigation. Numerical illustrations are given to demonstrate that the pattern of convergence depends on N in the search for D-optimality. The computations were conducted in R version 4.1.1 (2021). The result shows that the pattern of convergence differs with respect to the nature of the N-point design; even or odd, and to the degree of the two-variable response function; linear, interactive, and quadratic.***

Keywords: Cycling, Variance exchange process, D-optimality, Response function, Even and odd N-point designs

1. Introduction

The target of every designer is to construct a sequence of designs in which

$$\lim_{n\to\infty} \det M\left\{\left(\xi_N^{(n)}\right)\right\} = \det\left\{M\left(\xi_N^{(\cdot)}\right)\right\}, \quad (1)$$

where $\xi^{(\cdot)}$, is an *N*-point exact *D*-optimal design (Atkinson et al, 2007). Oftentimes the sequential search procedure for finding a *D*-optimal design in (1) is slowed down or almost difficult to achieve as a result of cycling (Ikpan and Nwobi, 2021). The search procedure for the construction of exact *D*-optimum designs starts with an initial design of size *N* selected from a sample space of *N* candidate points and proceeds by replacing points $\mathbf{x}_i$, in the design with minimum variances by those $\mathbf{x}_j$, with maximum variances from the candidate set, with the number of points *N* remaining fixed (Atkinson et al (2007) and Al Labadi (2015)).

The sequential search structure of the variance exchange process according to Pazman (1986), Atkinson and Donev (1992), Onukogu (1997), and Atkinson et al (2007), is based on the equivalence between *G*- and *D*-optimality which holds when the design measure is approximate or continuous design but not exact. By Atkinson, et al (2007), the Equivalence theorem for *D*- and *G*-Optimality states that continuous designs that are *D*-Optimum are also *D*-Optimum; in other words, they minimize the maximum over *X* of the variance

$$d(\mathbf{x},\xi) = \mathbf{x}^T M^{-1}(\xi)\mathbf{x} \qquad (2)$$

The choice of the points $\mathbf{x}_i$ and $\mathbf{x}_j$ in the iterative process depends on the variance of the predicted response at these points, the determinant of the information matrix, and the values of elements of its inverse.

Cycling, according to Ikpan & Nwobi (2021), is the inability of a monotone non-decreasing sequence to converge due to circular behavior of the variance points within and outside of a design matrix in an exchange process. When cycling occurs, a point in the current design with relatively low variance exchanged with a better variance point from the complement design, at a certain stage, becomes eligible at another stage to return to the design creating a steady or circular pattern of determinants of such designs. The effect of this is that

(i) The determinant $\det M\left(\xi_N^{(k)}\right)$ of the information matrix at a certain step of the exchange, the *k*th step, is equal to the determinant $\det M\left(\xi_N^{(k+1)}\right)$ of the information matrix of a later step, the $(k+1)$th step or even $\det M\left(\xi_N^{(k-1)}\right)$, of an earlier step, the $(k-1)$th step; and hence, does not have a defined maximum.
(ii) Convergence to *D*-optimality becomes slowed down, or almost impossible.

The pattern of convergence at the points of cycling depends on whether the initial design of size *N*is either randomly chosen even*N*-point designs or odd *N*-point designs. In this article, our target is to study the pattern of convergence when cycling occurs with respect to even and odd *N*point designs in a 2-variable response function.

2. Reference Literature

The appraisal of convergence patterns with respect to even and odd *N*-point designs when cycling occurs in a variance exchange process joins a rich literature of variance exchange algorithms in the construction of optimal exact designs. Our refenced works include the studies published by Cook and Nachtsheim (1980) – "A comparison of algorithms for constructing exact *D*-optimal designs". Technometrics[5]; Johnson and Nachtsheim (1983) – "Some guidelines for constructing exact *D*-optimal designs on convex design spaces". Technometrics[9]; Atkinson and Donev (1989) – "The construction of exact *D*-optimum experimental designs with application to blocking response surface designs". Biometrika[2].

Similar sequential search structure of the variance exchange process we referenced are according to Pazman (1986) – "Foundations of Optimum Experimental Design", Reidel[14]; Atkinson and Donev (1992) – "Optimum Experimental Designs", Oxford University Press[4]; Onukogu (1997) – "Foundations of Optimal Exploration of Response Surfaces", Exphrata Press[12]; Atkinson et al (2007) – "Optimum Experimental Designs, with SAS", Oxford University Press[5]; and Onukogu, and Chigbu (2002): Super Convergent Line Series in Optimal Designs of experiment and Mathematical Programming. A. P. Express Publishers[13]

Other works consulted are those of Eccleston and Jones (1980) – "Exchange and interchange procedures to search for optimal design", Journal of Royal Statistical Society[6]; Donev and Atkinson (1988), "An adjustment algorithm for the construction of exact D-optimum experimental designs", Technometrics[7]; Al Labadi (2015), "Some refinements on Fedorov's algorithms for constructing D-optimal designs", Brazilian Journal of Probability and Statistics[1]; Ikpan and Nwobi (2021), "Cycling in a Variance Exchange Algorithm: Its Influence and Remedy", Afrika Statistika[8].

3. Methods / Approach

Cycling is known to affect determinants in a variance exchange process. The effect on the determinants is seen to induce different patterns of convergence which in turn halts improvement to *D*-optimality of the process. The approach is to search the design space to ascertain the nature of convergence influenced by cycling.

3.1 CYCLING

As Ikpan and Nwobi (2021) explained, cycling occurs in an iterative process in accordance with the following:

(I) Set up an initial starting design, $\xi_N^{(1)}$, of $N > p$ points, from a set of candidate points, N, where *p*, is the number of parameters of the response function, and calculate the determinant of the resulting Information Matrix, $\left|M\left(\xi_N^{(1)}\right)\right|$.

(II) Compute the variances, $d\left(\mathbf{x}_i, \xi_N^{(1)}\right)$, of each of the *N*-points in the design $X_N^{(1)}$, and $d\left(\mathbf{x}_j, \xi_N^{(1)c}\right)$, of the complement design, $X_N^{(1)c}$, where

$$\xi_N^{(1)} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_v \\ \vdots \\ \mathbf{x}_N \end{pmatrix}, \text{ and } \xi_N^{(1)c} = \begin{pmatrix} \mathbf{x}_{N+1} \\ \mathbf{x}_{N+2} \\ \vdots \\ \mathbf{x}_w \\ \vdots \\ \mathbf{x}_{\bar{N}} \end{pmatrix}.$$

(III) Is $d\left(\mathbf{x}_v, \xi_N^{(1)}\right) > d\left(\mathbf{x}_w, \xi_N^{(1)}\right)$?

a) Yes, Stop

b) No, exchange $d\left(\mathbf{x}_v, \xi_N^{(1)}\right)$, the point with minimum variance in the design, with $d\left(\mathbf{x}_w, \xi_N^{(1)}\right)$, the point with maximum variance in the complement design, and define a new design measure, $\xi_N^{(2)}$.

c) Compute the Determinant $\left|M\left(\xi_N^{(2)}\right)\right|$ and notice that $\left|M\left(\xi_N^{(2)}\right)\right| > \left|M\left(\xi_N^{(1)}\right)\right|$

d) Repeat steps (II) and (III) until the kth step.

If

$$d\left(\mathbf{x}_w, \xi_N^{(k)}\right) \le d\left(\mathbf{x}_v, \xi_N^{(k-1)}\right) \text{ and } \left|M\left(\xi_N^{(k)}\right)\right| > \left|M\left(\xi_N^{(k-1)}\right)\right|,$$

then the system has converged to the desired optimum. Cycling occurs otherwise, if

$$d\left(\mathbf{x}_w, \xi_N^{(k)}\right) > d\left(\mathbf{x}_v, \xi_N^{(k-1)}\right) \text{ and }$$
$$\left|M\left(\xi_N^{(k)}\right)\right| \le \left|M\left(\xi_N^{(k-1)}\right)\right| X_N^{(1)c}.$$

Cycling resulting in such an iterative process is often with no improvement in the value of the determinant of the information matrix.

3.2 Convergence Pattern

The influence of cycling in a variance exchange process to a great extent, induces the pattern of convergence and this depends on the size, *N*, of the design measure, ξ_N, whether it is even or odd. The following statements illustrate the pattern of convergence.

Statement 1.0

Suppose $\xi_N^{(k)}$, be a design measure of an *N*-point design at the point of cycling, the kth step of an exchange process searching for exact *D*-optimum design. If *N* is even in a 2-variable linear and higher (interactive) order effects design, then

$$\left|M\left(\xi_N^{(k)}\right)\right| = \left|M\left(\xi_N^{(k-2)}\right)\right| \quad (3)$$

where $\xi_N^{(k-2)}$, is the measure of a design, two steps backward.

Corollary 1.0

Suppose $\xi_N^{(k)}$, be a design measure of an *N*-point design at the point of cycling, the kth step of an exchange process searching for exact *D*-optimum design. If *N* is even in a 2-variable linear and higher (mixed) order effects design, then

$$\left.\begin{aligned} d\left(\mathbf{x}_v, \xi_N^{(k)}\right) &= d\left(\mathbf{x}_v, \xi_N^{(k-2)}\right) \\ d\left(\mathbf{x}_w, \xi_N^{(k)}\right) &= d\left(\mathbf{x}_w, \xi_N^{(k-2)}\right) \end{aligned}\right\} \quad (4)$$

where $d\left(\mathbf{x}_v, \xi_N^{(\cdot)}\right)$ and $d\left(\mathbf{x}_w, \xi_N^{(\cdot)}\right)$, are respectively the minimum and maximum variances of the designs.

Proof:
Let

$$\xi_N^{(k-2)}=\begin{pmatrix}\mathbf{x}_{1k-2}\\ \mathbf{x}_{2k-2}\\ \vdots\\ \mathbf{x}_{vk-2}\\ \vdots\\ \mathbf{x}_{Nk-2}\end{pmatrix};\ \xi_N^{(k-1)}=\begin{pmatrix}\mathbf{x}_{1k-1}\\ \mathbf{x}_{2k-1}\\ \vdots\\ \mathbf{x}_{vk-1}\\ \vdots\\ \mathbf{x}_{Nk-1}\end{pmatrix};\text{ and }\xi_N^{(k)}=\begin{pmatrix}\mathbf{x}_{1k}\\ \mathbf{x}_{2k}\\ \vdots\\ \mathbf{x}_{vk}\\ \vdots\\ \mathbf{x}_{Nk}\end{pmatrix};\quad (5)$$

the design measures of a variance exchange process corresponding to the $(k-2)$th, $(k-1)$th and the kth steps respectively and let $\xi_N^{(k)}$ be the design measure at the point of cycling.

Let also $d\left(\mathbf{x}_v,\xi_N^{(k)}\right)$ and $d\left(\mathbf{x}_w,\xi_N^{(k)}\right)$, be defined for $X_N^{(k)}$ and $X_N^{(1)c}$, the current and complement design matrices at the point of cycling, respectively as follows

$$\left.\begin{aligned} d\left(\mathbf{x}_v,\xi_N^{(k)}\right) &= \min_{\mathbf{x}\in X_N^{(k)}} \mathbf{x}_i' M^{-1}\left(\xi_N^{(k)}\right)\mathbf{x}_i \\ d\left(\mathbf{x}_w,\xi_N^{(k)}\right) &= \max_{\mathbf{x}\in X_N^{(k)c}} \mathbf{x}_j' M^{-1}\left(\xi_N^{(k)}\right)\mathbf{x}_j \end{aligned}\right\}\quad (6)$$

Similarly, define $d\left(\mathbf{x}_v,\xi_N^{(k-2)}\right)$ and $d\left(\mathbf{x}_w,\xi_N^{(k-2)}\right)$, respectively for $X_N^{(k-2)}$ and $X_N^{(k-2)c}$, the design and its complement matrices at the $(k-2)$th step of the exchange process.

$$\left.\begin{aligned} d\left(\mathbf{x}_v,\xi_N^{(k-2)}\right) &= \min_{\mathbf{x}\in X_N^{(k-2)}} \mathbf{x}_i' M^{-1}\left(\xi_N^{(k-2)}\right)\mathbf{x}_i \\ d\left(\mathbf{x}_w,\xi_N^{(k-2)}\right) &= \max_{\mathbf{x}\in X_N^{(k-2)c}} \mathbf{x}_j' M^{-1}\left(\xi_N^{(k-2)}\right)\mathbf{x}_j \end{aligned}\right\}\quad (7)$$

If N, the sizeof the design, is even, and

(i) $d\left(\mathbf{x}_v,\xi_N^{(k)}\right)=d\left(\mathbf{x}_v,\xi_N^{(k-2)}\right)$, and

(ii) $d\left(\mathbf{x}_w,\xi_N^{(k)}\right)=d\left(\mathbf{x}_w,\xi_N^{(k-2)}\right)$, by (6) and (7);

then.

$$\left|M\left(\xi_N^{(k)}\right)\right|=\left|M\left(\xi_N^{(k-2)}\right)\right|,$$

and the design, 2-factor and of linear or interactive order (QED).

Statement 2

Suppose $\xi_N^{(k)}$, be a design measure of an N-point design at the point of cycling, the kth step of an exchange process searching for exact D-optimum design. If N is odd in a 2-variable linear and higher (mixed) order effects design, then

$$\left|M\left(\xi_N^{(k)}\right)\right|=\left|M\left(\xi_N^{(k-1)}\right)\right|\quad (8)$$

where $\xi_N^{(k-1)}$, is the measure of a design, one step backward.

Corollary 2

Suppose $\xi_N^{(k)}$, be a design measure of an N-point design at the point of cycling, the kth step of an exchange process searching for exact D-optimum design. If N is odd in a 2-variable linear and higher (interactive) order effects designs, then

$$\left.\begin{aligned} d\left(\mathbf{x}_v,\xi_N^{(k)}\right) &= d\left(\mathbf{x}_v,\xi_N^{(k-1)}\right) \\ d\left(\mathbf{x}_w,\xi_N^{(k)}\right) &= d\left(\mathbf{x}_w,\xi_N^{(k-1)}\right) \end{aligned}\right\}\quad (9)$$

where $d\left(\mathbf{x}_v,\xi_N^{(\cdot)}\right)$ and $d\left(\mathbf{x}_w,\xi_N^{(\cdot)}\right)$, are respectively the minimum and maximum variances of the designs.

Proof

Let $\xi_N^{(k-1)}$ and $\xi_N^{(k)}$ be as defined in equation (5).Define $d\left(\mathbf{x}_v,\xi_N^{(k-1)}\right)$, and $d\left(\mathbf{x}_w,\xi_N^{(k-1)}\right)$ respectively for $X_N^{(k-1)}$ and $X_N^{(k-1)c}$, the design and its complement matrices, at the kth step of the exchange process, as follows

$$\left.\begin{aligned} d\left(\mathbf{x}_v,\xi_N^{(k-1)}\right) &= \min_{\mathbf{x}\in X_N^{(k-1)}} \mathbf{x}_i' M^{-1}\left(\xi_N^{(k-1)}\right)\mathbf{x}_i \\ d\left(\mathbf{x}_w,\xi_N^{(k-1)}\right) &= \max_{\mathbf{x}\in X_N^{(k-1)c}} \mathbf{x}_j' M^{-1}\left(\xi_N^{(k-1)}\right)\mathbf{x}_j \end{aligned}\right\}\quad (10)$$

If the size of the N-point design, is odd, andby (6) and (10)

(i) $d\left(\mathbf{x}_v,\xi_N^{(k)}\right)=d\left(\mathbf{x}_v,\xi_N^{(k-1)}\right)$, and

(ii) $d\left(\mathbf{x}_w,\xi_N^{(k)}\right)=d\left(\mathbf{x}_w,\xi_N^{(k-1)}\right)$;

then.

$$\left|M\left(\xi_N^{(k)}\right)\right|=\left|M\left(\xi_N^{(k-1)}\right)\right|,$$

and the design, 2-factor and of linear or interactive order (QED).

Statement 3

Suppose $\xi_N^{(k)}$, be an N-point design measure at the point of cycling, the kth step of an exchange process searching for exact D-optimum design. Whether N is even or odd, if the 2-variable design, ξ_N, is a quadratic effects design, then

$$\left|M\left(\xi_N^{(k)}\right)\right|=\left|M\left(\xi_N^{(k-)}\right)\right|\quad (11)$$

where $\xi_N^{(k-)}$, is the measure of a design, a fraction of one or two steps backwards.

Corollary 3

Suppose $\xi_N^{(k)}$, be an N-point design measure at the point of cycling, the kth step of an exchange process searching for exact D-optimum design, and $\xi_N^{(k-)}$, a design at a fraction of one or two steps backwards in the sequence. Whether $N$is even or odd, if the 2-variable design, ξ_N, is a quadratic effects design, then

$$\left.\begin{aligned} d\left(\mathbf{x}_v,\xi_N^{(k)}\right) &= d\left(\mathbf{x}_v,\xi_N^{(k-)}\right) \\ d\left(\mathbf{x}_w,\xi_N^{(k)}\right) &= d\left(\mathbf{x}_w,\xi_N^{(k-)}\right) \end{aligned}\right\}\quad (12)$$

where $d\left(\mathbf{x}_v,\xi_N^{(\cdot)}\right)$ and $d\left(\mathbf{x}_w,\xi_N^{(\cdot)}\right)$, are the minimum and maximum variances of the respective designs

Proof

Consider equation (3), and define $d\left(\mathbf{x}_v,\xi_N^{(k-)}\right)$ and $d\left(\mathbf{x}_w,\xi_N^{(k-)}\right)$ respectively for $X_N^{(k-)}$ and $X_N^{(k-)c}$, the design and its complement matrices, at a fraction of one or two steps backwards, of the exchange process, as follows

$$\left.\begin{aligned} d\left(\mathbf{x}_v, \xi_N^{(k-)}\right) &= \min_{\mathbf{x}\in X_N^{(k-)}} \mathbf{x}_i' M^{-1}\left(\xi_N^{(k-)}\right)\mathbf{x}_i \\ d\left(\mathbf{x}_w, \xi_N^{(k-)}\right) &= \max_{\mathbf{x}\in X_N^{(k-)c}} \mathbf{x}_j' M^{-1}\left(\xi_N^{(k-)}\right)\mathbf{x}_j \end{aligned}\right\} \quad (13)$$

Whether the size of the N-point design is even or odd, andby (6) and (13)

(i) $d\left(\mathbf{x}_v, \xi_N^{(k)}\right) = d\left(\mathbf{x}_v, \xi_N^{(k-1)}\right)$, and

(ii) $d\left(\mathbf{x}_w, \xi_N^{(k)}\right) = d\left(\mathbf{x}_w, \xi_N^{(k-1)}\right)$;

then

$$\left|M\left(\xi_N^{(k)}\right)\right| = \left|M\left(\xi_N^{(k-)}\right)\right|,$$

and the design, 2-factor and of quadratic order effect. (QED)

4. Results and Discussion

An experimental space of two variables in linear, mixed and quadratic response functions generates two sets of even and odd N-point designs including 8 and10, and 9 and 11-point designs. The linear components in the different N-point designs have two design alternatives each. The statistical model for two-variable response function, is

$$f(x_1, x_2) = a_0 + \sum_{i=1}^{2} a_i x_i + \sum_{i=1}^{2} a_{ii} x_i^2 + \sum_{i<j}\sum a_{ij} x_i x_j + e. \quad (14)$$

and the experimental set is

$$\tilde{X} = \{x_1, x_2;\ x_1 = -2, -1, 0, 1, 2,\ x_2 = -1, -0.5, 0.5, 1\},$$

$$E(e) = 0 \text{ and } \mathrm{Var}(e) = \sigma_e^2$$

Computations are conducted in R version 4.1.1 (R Core Team (2021))

4.1 The even N-Point Designs

The initial and complement extended design matrices for the 8 and 10-point linear alternatives and interactive effect designs, are given respectively as

$$X_8^{(1)} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & -2 & -0.5 \\ 1 & -1 & 0.5 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0.5 \\ 1 & 2 & -0.5 \\ 1 & 2 & 1 \end{pmatrix}, X_8^{(1)c} = \begin{pmatrix} 1 & -2 & 0.5 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -0.5 \\ 1 & -1 & 1 \\ 1 & 0 & -0.5 \\ 1 & 0 & 0.5 \\ 1 & 1 & -1 \\ 1 & 1 & -0.5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & 0.5 \end{pmatrix}; X_{8(II)}^{(1)} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & -2 & 0.5 \\ 1 & -1 & -0.5 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -0.5 \\ 1 & 2 & 1 \end{pmatrix}, X_{8(II)}^{(1)c} = \begin{pmatrix} 1 & -2 & -0.5 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 0.5 \\ 1 & 0 & -0.5 \\ 1 & 0 & 0.5 \\ 1 & 0 & 1 \\ 1 & 1 & 0.5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & -0.5 \\ 1 & 2 & 0.5 \end{pmatrix},$$

$$X_8^{(1)} = \begin{pmatrix} 1 & -2 & -0.5 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -0.5 & 0.5 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -0.5 & 0 \\ 1 & 1 & -0.5 & -0.5 \\ 1 & 2 & -0.5 & -1 \\ 1 & 2 & 1 & 2 \end{pmatrix}, X_8^{(1)c} = \begin{pmatrix} 1 & -2 & -1 & 2 \\ 1 & -2 & 0.5 & -1 \\ 1 & -2 & 1 & -2 \\ 1 & -1 & 0.5 & -0.5 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0.5 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 2 & 0.5 & 1 \end{pmatrix}; X_{10}^{(1)} = \begin{pmatrix} 1 & -2 & -0.5 & 1 \\ 1 & -2 & 0.5 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -0.5 & 0.5 \\ 1 & 0 & -0.5 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 2 & 0.5 & 1 \end{pmatrix}, X_{10}^{(1)} = \begin{pmatrix} 1 & -2 & -1 & 2 \\ 1 & -2 & 1 & -2 \\ 1 & -1 & 0.5 & -0.5 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0.5 & 0 \\ 1 & 1 & -0.5 & -0.5 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & 2 & -0.5 & -1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

and

$$X_{10}^{(1)} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & -2 & 0.5 \\ 1 & -1 & -0.5 \\ 1 & -1 & 1 \\ 1 & 0 & 0.5 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -0.5 \\ 1 & 2 & -0.5 \\ 1 & 2 & 0.5 \end{pmatrix}, X_{10}^{(1)c} = \begin{pmatrix} 1 & -2 & -0.5 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 0.5 \\ 1 & 0 & -1 \\ 1 & 0 & -0.5 \\ 1 & 1 & 0.5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & 1 \end{pmatrix}; X_{10(II)}^{(1)} = \begin{pmatrix} 1 & -2 & -0.5 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 0.5 \\ 1 & 0 & -1 \\ 1 & 0 & -0.5 \\ 1 & 1 & 0.5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & 1 \end{pmatrix}, X_{10(II)}^{(1)c} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & -2 & 0.5 \\ 1 & -1 & -0.5 \\ 1 & -1 & 1 \\ 1 & 0 & 0.5 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -0.5 \\ 1 & 2 & -0.5 \\ 1 & 2 & 0.5 \end{pmatrix}$$

Table 1 gives the summary of the convergence pattern for the even linear *N*-point designs

Table 1: Pattern of convergence for the linear and interactive,8 and 10-point designs

$\xi_N^{(k)}$	$\lvert M(\xi_8^{(k)})\rvert$			$\lvert M(\xi_{10}^{(k)})\rvert$		
	Linear Order Effect		Interactive Effect	Linear Order Effect		Interactive Effect
	I	II		I	II	
1	592	675	3746.25	1090	1390	10395
2	886.5	911.5	8766	1461	1739	22834.69
3	1224	1163.5	15450.75	1774.5	2006	32832
4	1262.5	1294	24336	2100.5	2251.5	50544
5	1300	1326	26316	2349	2349	75688.87
6	1326	1294	24336	2352	2352	50544
7	1300			2349	2349	

The initial and complement design matrices for quadratic effects, two-variable 8 and 10-point designs, are as follows

$$X_8^{(1)} = \begin{pmatrix} 1 & -2 & -0.5 & 1 & 4 & 0.25 \\ 1 & -2 & 0.5 & -1 & 4 & 0.25 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -0.5 & 0 & 0 & 0.25 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 2 & -0.5 & -1 & 4 & 0.25 \\ 1 & 2 & 1 & 2 & 4 & 1 \end{pmatrix}, X_8^{(1)} = \begin{pmatrix} 1 & -2 & -1 & 2 & 4 & 1 \\ 1 & -2 & 1 & -2 & 4 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -0.5 & 0.5 & 1 & 0.25 \\ 1 & -1 & 0.5 & -0.5 & 1 & -0.25 \\ 1 & 0 & 0.5 & 0 & 0 & 0.25 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & -0.5 & -0.5 & 1 & 0.25 \\ 1 & 1 & 0.5 & 0.5 & 1 & 0.25 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 & 4 & 1 \\ 1 & 2 & 0.5 & 1 & 4 & 0.25 \end{pmatrix}$$

and

$$X_{10}^{(1)} = \begin{pmatrix} 1 & -2 & -1 & 2 & 4 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -0.5 & 0.5 & 1 & 0.25 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -0.5 & 0 & 0 & 0.25 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & -0.5 & -0.5 & 1 & 0.25 \\ 1 & 1 & 0.5 & 0.5 & 1 & 0.25 \\ 1 & 2 & -1 & -2 & 4 & 1 \\ 1 & 2 & 0.5 & 1 & 4 & 0.25 \end{pmatrix}, X_{10}^{(1)c} = \begin{pmatrix} 1 & -2 & -0.5 & 2 & 4 & 0.25 \\ 1 & -2 & 0.5 & -1 & 4 & 0.25 \\ 1 & -2 & 1 & -2 & 4 & 1 \\ 1 & -1 & 0.5 & -0.5 & 1 & 0.25 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 0.5 & 0 & 0 & 0.25 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -0.5 & -1 & 1 & 0.25 \\ 1 & 2 & 1 & 2 & 4 & 1 \end{pmatrix}$$

4.2The odd *N*-Point Designs

The initial and complement extended design matrices for the 9, and 11-point linearalternatives and mixed component designs are given respectively as

$$X_9^{(1)}=\begin{pmatrix}1&-2&-1\\1&-2&-0.5\\1&-1&-1\\1&-1&0.5\\1&0&-1\\1&1&-0.5\\1&1&1\\1&2&0.5\\1&2&1\end{pmatrix},X_9^{(1)c}=\begin{pmatrix}1&-2&0.5\\1&-2&1\\1&-1&-0.5\\1&-1&1\\1&0&-0.5\\1&0&0.5\\1&0&1\\1&1&-1\\1&1&0.5\\1&2&-1\\1&2&-0.5\end{pmatrix};X_{9(II)}^{(1)}=\begin{pmatrix}1&-2&-0.5\\1&-2&1\\1&-1&-0.5\\1&0&-1\\1&0&0.5\\1&1&-1\\1&1&0.5\\1&2&-0.5\\1&2&1\end{pmatrix},X_{9(II)}^{(1)c}=\begin{pmatrix}1&-2&-1\\1&-2&0.5\\1&-1&-1\\1&-1&0.5\\1&-1&1\\1&0&-0.5\\1&0&1\\1&1&-0.5\\1&1&1\\1&2&-1\\1&2&0.5\end{pmatrix}$$

$$X_9^{(1)}=\begin{pmatrix}1&-2&-1&2\\1&-2&-0.5&1\\1&-1&-1&1\\1&-1&0.5&-0.5\\1&0&-1&0\\1&1&-0.5&-0.5\\1&1&1&1\\1&2&0.5&1\\1&2&1&2\end{pmatrix},X_9^{(1)c}=\begin{pmatrix}1&-2&0.5&-1\\1&-2&1&-2\\1&-1&-0.5&0.5\\1&-1&1&-1\\1&0&-0.5&0\\1&0&0.5&0\\1&0&1&0\\1&1&-1&-1\\1&1&0.5&0.5\\1&2&-1&-2\\1&2&-0.5&-1\end{pmatrix};X_{11}^{(1)}=\begin{pmatrix}1&-2&-1&2\\1&-2&-0.5&1\\1&-1&-0.5&0.5\\1&-1&0.5&-0.5\\1&-1&1&-1\\1&0&-1&0\\1&0&0.5&0\\1&1&-0.5&-0.5\\1&1&0.5&0.5\\1&1&1&1\\1&2&-1&-2\end{pmatrix},X_{11}^{(1)c}=\begin{pmatrix}1&-2&0.5&-1\\1&-2&1&-2\\1&-1&-1&1\\1&0&-0.5&0\\1&0&1&0\\1&1&-1&-1\\1&2&-0.5&-1\\1&2&0.5&1\\1&2&1&2\end{pmatrix}$$

and

$$X_{11}^{(1)}=\begin{pmatrix}1&-2&-1\\1&-2&-0.5\\1&-2&0.5\\1&-2&1\\1&-1&-1\\1&-1&-0.5\\1&-1&0.5\\1&-1&1\\1&0&-1\\1&0&-0.5\\1&0&0.5\end{pmatrix},X_{11}^{(1)c}=\begin{pmatrix}1&0&1\\1&1&-0.5\\1&1&0.5\\1&1&-1\\1&1&-0.5\\1&2&-1\\1&2&-0.5\\1&2&0.5\\1&2&1\end{pmatrix};\quad X_{11(II)}^{(1)}=\begin{pmatrix}1&-2&0.5\\1&-2&1\\1&-1&-1\\1&-1&-0.5\\1&0&-1\\1&0&1\\1&1&-0.5\\1&1&0.5\\1&2&-1\\1&2&-0.5\\1&2&0.5\end{pmatrix},X_{11(II)}^{(1)}=\begin{pmatrix}1&-2&-1\\1&-2&-0.5\\1&-1&0.5\\1&-1&1\\1&0&-0.5\\1&0&0.5\\1&1&-1\\1&1&1\\1&2&1\end{pmatrix}$$

Table 2 gives the summary of the convergence pattern for the odd *N*-point designs

Table 2: Pattern of convergence for the 9 and11-point linear and mixed component designs,

$\xi_N^{(k)}$	$\left\|M\left(\xi_9^{(k)}\right)\right\|$			$\left\|M\left(\xi_{11}^{(k)}\right)\right\|$		
	Linear Order Effect		Interactive Effect	Linear Order Effect		Interactive Effect
	I	II		I	II	
1	619.00	887.75	4258.50	474	1545.25	10435.50
2	1061.00	1116.00	11976.50	1203	2071.25	22698.00
3	1484.75	1392.75	27102.87	1974	2605.25	42623.13
4	1709.75	1608.00	32814.00	2501	2854.00	49428.00
5	1725.00	1806.00	36000.00	2926	2926.00	57011.62
6	1806.00	1806.00	36972.00	3204	3024.00	63063.00
7	1806.00		36972.00	3204	3024.00	68688.00
8						68688.00

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The initial and complement extended design matrices for higher order (quadratic), two-variable 9 and 11-point Designs, are given respectively as

$$X_9^{(1)} = \begin{pmatrix} 1 & -2 & -1 & 2 & 4 & 1 \\ 1 & -2 & -0.5 & 1 & 4 & 0.25 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & -0.5 & 0 & 0 & 0.25 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & -0.5 & -0.5 & 1 & 0.25 \\ 1 & 1 & 0.5 & 0.5 & 1 & 0.25 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 4 & 1 \end{pmatrix}, \quad X_9^{(1)c} = \begin{pmatrix} 1 & -2 & 0.5 & -1 & 4 & 0.25 \\ 1 & -2 & 1 & -2 & 4 & 1 \\ 1 & -1 & -0.5 & 0.5 & 1 & 0.25 \\ 1 & -1 & 0.5 & -0.5 & 1 & 0.25 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0 & 0 & 0.25 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 2 & -1 & -2 & 4 & 1 \\ 1 & 2 & -0.5 & -1 & 4 & 0.25 \\ 1 & 2 & 0.5 & 1 & 4 & 0.25 \end{pmatrix}$$

and

$$X_{11}^{(1)} = \begin{pmatrix} 1 & -2 & -1 & 2 & 4 & 1 \\ 1 & -2 & -0.5 & 1 & 4 & 0.25 \\ 1 & -1 & -0.5 & 0.5 & 1 & 0.25 \\ 1 & -1 & 0.5 & -0.5 & 1 & 0.25 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0 & 0 & 0.25 \\ 1 & 1 & -0.5 & -0.5 & 1 & 0.25 \\ 1 & 1 & 0.5 & 0.5 & 1 & 0.25 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 & 4 & 1 \end{pmatrix}, \quad X_{11}^{(1)c} = \begin{pmatrix} 1 & -2 & 0.5 & -1 & 4 & 0.25 \\ 1 & -2 & 1 & -2 & 4 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & -0.5 & 0 & 0 & 0.25 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 2 & -0.5 & -1 & 4 & 0.25 \\ 1 & 2 & 0.5 & 1 & 4 & 0.25 \\ 1 & 2 & 1 & 2 & 4 & 0.25 \end{pmatrix}$$

Table 3 gives the summary of the convergence pattern for both the even and odd quadratic*N*-point designs

Table 3: Pattern of convergence for the 8, 9, 10 and 11-point quadratic effect designs

$\xi_N^{(k)}$	Even *N*-points		Odd*N*-points	
	$\left\|M\left(\xi_8^{(k)}\right)\right\|$	$\left\|M\left(\xi_{10}^{(k)}\right)\right\|$	$\left\|M\left(\xi_9^{(k)}\right)\right\|$	$\left\|M\left(\xi_{11}^{(k)}\right)\right\|$
1	71721.0	53230.5.0	8354.25	254229.8
2	153090.0	441414.6	46379.25	643248.0
3	261528.8	910818.0	304618.50	1472981.0
4	324900.0	1329552.0	645462.00	1827636.0
5	315000.0	1521792.0	553797.00	2033151.0
6	352368.0	1475712.0	774864.00	1992587.0
7	319023.0		657252.00	

4.3 Discussion of Results

The results of Table 1, shows thatthe determinants of the information matrices of the 8 and 10-point designs at the points of cycling are equal to the ones two places before them in the exchange process. This is in line with Statement 1.0 which states that an even *N*-point designof linear and interactive order effects, at the point of cycling, the kth step of an exchange process searching for exact *D*-optimum design, has determinant equal to that of the $(k-2)$th step.

The results of Table 2, shows thatthe determinants of the information matrices of the 9 and 11-point designs at the points of cycling are equal to the ones next to them in the exchange process. This is in line with Statement 2.0 which states that an odd*N*-point designof linear and interactive order effects, at the point of cycling, the kth step of an exchange process searching for exact *D*-optimum design, has determinant equal to that of the $(k-1)$th step.

The results of Table 3, shows that the determinants of the information matrices for all of the quadratic*N*-point designs at the points of cycling fall betweenthose of one and two places before them in the exchange process. This is in line with Statement 3.0 which states that an even or odd *N*-point designof quadrat order effect, at the point of cycling, the kth step of an exchange process searching for exact *D*-optimum design, has determinantlyingone and two steps before it, the $(k-)$th step.

5 Conclusion

From the results of the analysis, we may conclude thatthe pattern of convergence depends on the nature of the *N*- point design, even or odd, and on the degree of the response variable, linear, interactive or quadratic.For even, linear and interactive*N*-point designs, the determinant of the information matrix and variances of points at the point of cycling, are equal to the determinant of the information matrix and variances of points two places before it in the sequence. For odd,linearand mixed *N*-point designs, the determinant of the information matrix and variances of points at the point of cycling,are equal to the determinant of the information matrix and variances of points one place before it in the exchange process. All of quadratic *N*-point designs,even and odd, havedeterminants of the information matrix and variances of points at the point of cycling, equal to those of a fraction of one or two places before them in the exchange process.

6 Future Scope

The first degree and the mixed polynomials follow a regular pattern of determinants cycling about specific values giving a different picture of irregular patterns with the second order degree, which cycles about non-specific values. Emphasis on the future research direction should be focused on cycling patterns with respect to polynomials of the third degree.

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