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On Convexity of Fuzzy Sets and Fuzzy Relations

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Abstract: ***This paper explores the convexity of fuzzy sets and fuzzy relations defined on the set of real numbers and their cartesian product respectively. Using the concept of α-cut, we establish that a fuzzy set is convex if and only if the α-cut of the fuzzy set is convex. The paper further extends this concept to fuzzy relations. The findings have significant implications for the application of convexity in various fields.***

Keywords: Fuzzy sets, Convex fuzzy sets, Fuzzy relations, Convex fuzzy relations, α-cuts

1. Introduction

Many researchers have conducted numerous studies in the area of convexity by examining the significance of its numerous challenges in the real world. Prof. Zadeh first time coined the terms fuzzy set, convex fuzzy set, fuzzy relation, $\alpha - cut$ [1] in 1965.A fuzzy set is a function $F: R \rightarrow [0, 1]$, called as membership function and $F(x)$ is called membership grade at x in $F; x \in R$; values of membership grades lie in [0,1]. Fuzzy relation S is a fuzzy set defined on the cartesian product of the set of real numbers.

Xinmin Yang [2], and Syau [5] proposed the notion of closed fuzzy sets and convex fuzzy sets and studied the connection between them. Nadaban and Dzitac [4] defined the convex fuzzy relation in their paper and established special types of fuzzy relations. Chen-Wei-Xu [6] introduced new fuzzy relations from old ones and proved results on the convexity of fuzzy relations. In [1], a definition of the convexity of fuzzy set using the notion of α −cut is given. In this paper we will study that fuzzy set F is convex if and only if α −cut of fuzzy set, F_α is convex;$\forall \alpha \in (0, 1]$. We will extend the convexity of a fuzzy Set with respect to α −cut to Fuzzy Relation and try to prove that fuzzy relation S is convex if and only ifα −cut of the fuzzy relation, S_α is convex; $\forall \alpha \in (0, 1]$. We will study that α −cut of a fuzzy set is an interval if F is a convex fuzzy set and study its connection with a strongly convex fuzzy set and strictly convex fuzzy set.

2. Preliminaries

Throughout this paper, F denotes fuzzy set defined on R $and\, S$ denotes fuzzy relation defined on R^2 Here are some definitions that will be useful in this paper.

2.1 Definition [6]:

A fuzzy set F defined on R is a function; $F: R \rightarrow [0,1]$ is called as membership function and $F(x)$ is called membership grade of $F\ at\ x$.

2.2 Definition [7]

A Fuzzy relation$S$ is a fuzzy set defined on the cartesian product of crisp sets$X_1 \times X_2 \times X_3 \times \ldots . . \times X_n$ where tuples $(x_1, x_2, x_3, \ldots . . , x_n)$ that may have varying degree of membership value is usually represented by a real number for closed interval [0,1] and indicate the strength of the present relation between elements of the topic. Consider $S: X \times Y \rightarrow [0,1]$ then the fuzzy relation on $X \times Y$ denoted by S or $S(x, y)$is defined as the set $S(X, Y) = \{((x, y), S(x, y))/(x, y) \in X \times Y\}$. Where $S(x, y)$ is the strength of the relation in two variables called membership function. It gives the degree of membership of the ordered pair $(x, y)\ in\ X \times Y$ a real number in the interval [0,1].

2.3Definition [3]:

S be fuzzy relation on $X \times Y$. Then S is convex if and only if

$$S\big(\mu(x_1, y_1) + (1 - \mu)(x_2, y_2)\big) \geq S(x_1, y_1) \wedge S(x_2, y_2)\ \ \forall\, (x_1, y_1), (x_2, y_2) \in X \times Y\ and\ \mu \in [0,1].$$

2.4Definition [6]:

Let S be a fuzzy relation defined on $X \times Y$ and α be such that $0 < \alpha \leq 1$.Thenα −cut of S is denoted by S_α is defined by$S_\alpha = \{(x, y) \in X \times Y / S(x, y) \geq \alpha\}$.

2.5 Definition [1]:

A be a fuzzy set defined on R and α be such that $0 < \alpha \leq 1$. Then α −cut of$F$, is denoted by $F_\alpha$and defined by$F_\alpha = \{x \in R / F(x) \geq \alpha\}$ is crisp set.

2.6 Definition [1]:

F be a fuzzy set defined on R. Then F is convex if and only if

$$F(\mu x_1 + (1 - \mu)x_2) \geq \min[F(x_1), F(x_2)]\,;\ \forall\, x_1, x_2 \in R\ and\ \mu \in (0, 1].$$

2.7 Definition [2]:

A fuzzy set F on R is said to be strongly convex fuzzy set if

$$F(\mu x_1 + (1 - \mu)x_2) > \min[F(x_1), F(x_2)];\qquad \forall\, x_1, x_2 \in R, x_1 \neq x_2 \text{ and } \mu \in (0,1).$$

2.8 Definition [2]:

A fuzzy set F on is said to be strictly convex fuzzy set if

$$F(\mu x_1 + (1 - \mu)x_2) > \min[F(x_1), F(x_2)];\qquad F(x_1) \neq F(x2),\ \forall x1, x2 \in R\ and\ \mu \in (0,1).$$

3. Main Results

3.1 Theorem

F be a fuzzy set defined on R then F is convex if and only if for every $\alpha \in (0, 1]$, F_α is convex.

Proof:
Suppose F is a convex fuzzy set defined on R.

To prove that F_α is convex, $\forall\alpha \in (0, 1]$.

Let, if possible, for some $\alpha \in (0, 1]$. F_α is not convex.

That is there exist $x, y \in F_\alpha$ such that,
$\mu x + (1-\mu)y \notin F_\alpha$; $\forall\mu \in [0,1]$.

Implies that, $F(\mu x + (1-\mu)y) < \alpha$
Since, $x \in F_\alpha, F(x) \geq \alpha$.
and $y \in F_\alpha, F(y) \geq \alpha$.

Given F is a convex fuzzy set.

Therefore $F(\mu x + (1-\mu)y) \geq \min(F(x), F(y))$
$\geq \min(\alpha, \alpha)$.
$= \alpha$.

By definition of α −cut of fuzzy set,
$\mu x + (1-\mu)y \in F_\alpha$.

A contradiction to our assumption that F_α is not convex,for some $\alpha \in (0, 1]$.

Therefore F_α is convex, $\forall\alpha \in (0, 1]$.

Conversely, suppose that F_α is convex;$\forall\alpha \in (0, 1]$.

To prove that F is Convex Fuzzy Set on R.

Let if possible, for some $x, y \in R$.

$F(\lambda x + (1-\lambda)y) < \min(F(x), F(y))$ (1)

Choose $\alpha = \min(F(x), F(y))$.

Then, clearly $x, y \in F_\alpha$ implies that, $F(\mu x + (1-\mu)y) \geq \alpha = \min(F(x), F(y))$.

A contradiction to (1).

Our assumption is wrong.

Therefore F is Convex Fuzzy Set on R.

3.2 Corollary:

F be a strongly (strictly) convex fuzzy set defined on F if and only if $F_{\alpha+}$ is convex; for all $\alpha \in (0, 1]$, where $F_{\alpha+}$ is strong α −cut of F.

3.3 Theorem

S be a fuzzy relation defined on $R \times R$ then S is convex if and only if for every $\alpha \in (0, 1]$;S_α is convex.

Proof
Suppose that S is convex fuzzy relation defined on $R \times R$.

To prove that S_α is convex.

Let if possible, for some $\alpha \in (0, 1]$. S_α is not convex.

Then $\exists(x_1, y_1), (x_2, y_2) \in S_\alpha$ such that $\mu(x_1, y_1) + (1-\mu)(x_2, y_2) \notin S_\alpha$; $\mu \in [0, 1]$.

Implies that $S\big(\mu(x_1, y_1) + (1-\mu)(x_2, y_2)\big) < \alpha$.

Since, $(x_1, y_1) \in S_\alpha; S(x_1, y_1) \geq \alpha$. (1)

and $(x_2, y_2) \in S_\alpha$; $S(x_2, y_2) \geq \alpha$.

Since S is convex fuzzy relation.

Consider,
$S(\mu(x_1, y_1) + (1-\mu)(x_2, y_2)) \geq$
$\min(S(x_1, y_1), S(x_2, y_2))$.
$\geq \min(\alpha, \alpha)$. (by 1)

$= \alpha$.

By definition of α −cut of fuzzy relation,

$\mu(x_1, y_1) + (1-\mu)(x_2, y_2) \in S_\alpha$.

A contradiction.

S_α is convex set, for every $\alpha \in (0, 1]$.

Conversely suppose that S_α is convex set for every $\alpha \in (0,1]$,

We will prove that , S is convex fuzzy relation on $R \times R$.

It is enough to prove that S is convex fuzzy set on $R \times R$.

Let $(x_1, y_1), (x_2, y_2) \in S_\alpha$.

Then,
$\mu(x_1, y_1) + (1-\mu)(x_2, y_2) \in S_\alpha$.
i.e.$S(\mu(x_1, y_1) + (1-\mu)(x_2, y_2)) \geq \alpha$. (2)
Consider left hand side of (2).
$S(\mu(x_1, y_1) + (1-\mu)(x_2, y_2))$.
$= S(\mu x_1 + (1-\mu)x_2, \mu y_1 + (1-\mu)y_2)$.
$= \min(S(\mu x_1 + (1-\mu)y_1), S(\mu y_1 + (1-\mu)y_2))$.
$\geq \alpha$. (by 2)

Since S_α is convex $\alpha \in (0, 1]$.

Choose $\alpha = \min(S(x_1, y_1), S(x_2, y_2))$.

Where $\alpha = S(x_1, y_1) < S(x_2, y_2)$. Then, $S\big(\mu(x_1, y_1) +$ *1−μx2,y2≥minSx1,y1,Sx2,y2.*

Therefore, S is convex fuzzy set on $R \times R$. i.e., S is a convex fuzzy relation on $R \times R$.

Theorem 3

S be a strongly convex fuzzy relation defined on R^2 then there exist unique element $(x_1, x_2\ )$ such that$S(x_1, x_2\ ) = \max\{S(y_1, y_2)/(y_1, y_2) \in R^2\}$.

Proof.

Suppose that there are two elements $\left(x_{1,}x_2\ \right)$ and (z_1, z_2) in R^2such that

$\alpha = S(x_1, x_2\ ) = S(z_1, z_2) = \max\{S(y_1, y_2)/(y_1, y_2) \in$ *R2}.*

where,$(x_1, x_2\ ) \neq (z_1, z_2)$.

$S_\alpha = \{(x_1, x_2), (z_1, z_2)\}$.

Given that S is strongly convex fuzzy relation on R^2. therefore S_α is convex; $\alpha \in (0, 1)$.

As $\left(x_{1,}x_2\ \right) \neq (z_1, z_2)$.

We have, $\mu(x_1, x_2) + (1 - \mu)(z_1, z_2) \notin S_\alpha$as S_α contains only two elements; for all $\alpha \in (0, 1)$ and $\mu \in (0,1)$.

Therefore S_α is not convex. a contradiction.

$\therefore \left(x_{1,}x_2\ \right) = (z_1, z_2)$.

Therefore, there exist unique element $\left(x_{1,}x_2\ \right)$ such that $S\left(x_{1,}x_2\right) = \max\{S(y_1, y_2)/(y_1, y_2) \in R^2\}$

Converse part of the above theorem is not true in general; we may take non convex fuzzy set and find unique maximum element.

Theorem 4

S is convex fuzzy relation defined on R^2 then for every $\alpha \in (0, 1]$, S_α is closed and connected.

Proof

First, we prove that S_α is closed set.

Let (x, y) be any limit point of Sα then there exists sequence $(x_n, y_n) \in S_\alpha$.

$(x_n, y_n) \to (x, y)\ \forall\ n \geq N$..

As $(x_n, y_n) \in S_\alpha$ we have $S(x_n, y_n)) \geq \alpha$. Therefore $\lim_{n\to\infty} S(x_n, y_n) \geq \alpha$.

(x, y) is limit point of S_α;(x, y) is either an interior point or boundary point of S_α.

Case 1:(x, y) is an interior point of S_α.then clearly $S(x, y) \geq \alpha$.

Case 2:(x, y) is boundary point of S_α. we can find $\in > 0$ such that $(x{-}\in, x{+}\in)$ is an interior point of S_α.

Given that S is convex fuzzy relation on R^2.

Then by definition of convexity of fuzzy relation, S_α is convex, for all $\alpha \in (0, 1]$.

For,(x_n, y_n), (x-∈,x+∈) $\in S_\alpha$; $\mu(x_n, y_n) + (1 - \mu)$(x−∈ *,x+∈∈Sα.*

$$\lim_{n\to\infty} \mu(x_n, y_n) + (1 - \mu)(x{-}\in, x{+}\in) \in S_\alpha.$$

$\lim_{n\to\infty} \mu(x_n, y_n) + \lim_{n\to\infty}(1 - \mu)(x{-}\in, x{+}\in) \in S_\alpha$.

$\mu(x, y) + (1 - \mu)(x{-}\in, y{-}\in) \in\ S_\alpha$.

$for\ \mu = 1, (x, y) \in S_\alpha$.

Therefore $S(x, y) \geq \alpha$.

Since (x, y) is an arbitrary limit point of Sα.

Therefore Sα is closed.

Now to prove that Sα is connected.

Let, if possible, it is not connected. Then there exist two non-empty, disjoint, open sets A and B such that S_α=A∪B and A∩B=∅.

We can choose $(x, y) \in A\ and\ (x', y') \in B$.

Then $\mu(x, y) + (1 - \mu)(x', y') \notin S_\alpha$.

S_α is not convex set. Contradiction. Therefore S_α is connected.

Converse of the above theorem is not true in general.

For example. $S(x, y) = 1\ if\ (x, y) \in \Big\{(x, y) \in \frac{R^2}{x^2} + y^2 \geq$ *1 and x2+y2≤4.*

$$= \frac{1}{2}\ if\ (x, y) \in R^2/x^2 + y^2 < 1\ and\ x^2 + y^2 > 4\}.$$

Then $S_1 = (x, y) \in \{(x, y) \in R^2/\ \ x^2\ + y^2 \geq 1\ and\ x^2 +$ *y2≤4.*

which is closed and connected but not convex.

Theorem 5

Let $f: X \to Y$ is continuous crisp function and $F \in f(X)$ fuzzy set is connected then $f(F)$ is connected.

Proof

Given that F is connected then every α −cut of F is connected subset of X.

By extension principle of fuzzy sets and Theorem 2.9 [8],

$$f(\alpha_F) \subseteq \alpha_{[f(F)]} \qquad (1)$$

Suppose that $f(F)$ is not connected then $f(F) = Z \cup T$; where Z and T are two non empty, disjoint, open subsets of $f(F)$.

Taking strongα −cut of both sets.

$\alpha +_{[f(F)]} = \alpha +_{[Z\cup S]}$.

$$f(\alpha +_F) = \alpha +_{[Z]]} \cup \alpha +_{[S]}\ . \qquad (2)$$

We know that continuous image of connected set is connected.

Thus, left hand side of equation (2) is connected. Therefore, right hand side must be connected butright-hand side of equation (2) is union of two non-empty disjoint sets hence disconnected.

Thus, is absurd. Contradiction to our assumption. Therefore $f(F)$ is connected.

Since $\alpha +_F$ is connected $f(\alpha +_F)$ is connected.

4. Conclusion

We demonstrated the relationship between convex fuzzy set (convex fuzzy relation) and $\alpha -$cut of fuzzy set (fuzzy relation). $\alpha -$cut is work as bridgebetween crisp sets and fuzzy sets. Because convexity has tremendous applications in various fields, it’s crucial to investigate it using a fuzzy method at several levels. We have used it to look into the convexity of fuzzy sets and fuzzy relations.

5. Future Scope

We can generalize convexity of fuzzy relation for n-dimensional space. Connectivity of fuzzy sets and fuzzy relations using the notion of $\alpha -$cut could be studied.

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