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Exploring Normal Covering Spaces: A Bridge between Algebraic Topology and Abstract Algebra

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Abstract: ***The aim of this study was to introduce normal covering spaces which are a special class of covering spaces. The theory of covering spaces is a relevant topic of Algebraic Topology. Here we give description about the certain properties of the normal covering spaces. A normal covering space is characterised by its isotropy subgroup which is a normal subgroup of the concerned covering space. The study of covering space is closely connected with that of Fundamental group. The detailed explanation of covering spaces is given in [3] & [12]. The structure of Fundamental Group is given in [2], [5], & [11]. The basic results of Abstract Algebra is obtained from [9], & [10]. Normal covering spaces are a class of covering spaces which play a vital role in the study of Algebraic Topology. Here we make use of the algebraic structure used in Abstract Algebra and applied it in Algebraic Topology. The study of this article will help the researchers to do collaborative research with other areas of Mathematics.***

Keywords: Fundamental group, covering spaces, fibre set, loop product, homotopy, isotropy subgroup, orbit, stabilizer

1. Introduction

The systematic study of Algebraic topology was initiated by French Mathematician Henri Poincare (1854-1912). Algebraic Topology was motivated by specific geometric problems involving paths, surfaces and geometry in Euclidean spaces. It describes the structure of a Topological space by associating it with a group or a sequence of groups. Here we give description about Normal covering spaces. By using Fundamental group, topological problems about spaces can be reduced to purely algebraic problems about groups and homomorphisms .Many basic topological questions about covering spaces can be reduced to purely algebraic questions about the fundamental groups of various spaces involved. The fundamental group is an instrumental in determining and classifying topological spaces which are covering spaces of given space. Here we study the normal covering spaces which are a special class of covering spaces. The comprehensive study of Algebraic Topology is obtained from [4] &,[7].

2. Preliminaries

In this section we shall give the preliminary results required for the further work of this article. This section includes some definitions and lemma that are essential for developing main results our work. We also include some results from Abstract Algebra and Algebraic Topology that are essential for this discussion.

The detailed study of group action of Abstract Algebra is obtained from [9] & [10]. For the basic Topological results, reader can use [6]. The basic results of Algebraic Topology is obtained from [3] & [8].

3. Main Results and Discussions

In this section, we include the main results of our discussion. The discussion includes the fundamental group action on fibre set and the properties of a normal covering space.

4. Fundamental Group Action on Fibre Set

4.1.1 Definition
For any space B, if (E , p) is a covering space of B, then set p^{-1} (b) consists of all e in E such that p (e) = b is called **fibre set.**

4.1.2 Defining the Fundamental group action on fibre set
The action of the fundamental group of a topological space on a fibre set is defined as follows

Let (E, p) be any covering space of a topological space B.

Then for $[\alpha] \in \pi(B, b)$ and for any $e \in p^{-1}(b)$, there exists a unique lifting path $\tilde{\alpha}$ in E with $\tilde{\alpha}$ (0) = e

Then p $\tilde{\alpha}$ (1) = (1) = b, so that $\tilde{\alpha}(1) \in p^{-1}(b)$
This gives an association of an element in p^{-1} (x) with the pair (e, [α]).

Thus we get a correspondence from $p^{-1}(b)\times\pi(B, b)\rightarrow\pi(B, b)$

The map so defined has the following properties.

For [α], [β] in π (B, b) and $e \in p^{-1}(b)$

(e. [α]). [β] = $\tilde{\beta}$ (1) where $\tilde{\beta}$ is the path in E with initial point

e. [α] = $\tilde{\alpha}$ (1) and such that p$\tilde{\beta}$ = β

Now $\tilde{\alpha}$ and $\tilde{\beta}$ are paths in E such that $\tilde{\alpha}$ (1) = $\tilde{\beta}$ (0)

Hence the product $\tilde{\alpha} * \tilde{\beta}$ is defined in E such that

Now $\tilde{\alpha} * \tilde{\beta}$ (0) = $\tilde{\alpha}$ (0) = e

Thus $\tilde{\alpha} * \tilde{\beta}$ is a path in E with initial point e such that p ($\tilde{\alpha} * \tilde{\beta}$) = α * β

So by definition, e. ([α]. [β]) = e. ([α* β])

= $(\tilde{\alpha} * \tilde{\beta})$ (1)

= $\tilde{\beta}$ (1)

Hence we get (e. [α]). [β] = e. ([α]. [β])

The identity element of the fundamental group π (B, b) is $[C_b]$ where C_b is the constant loop defined by C_b (t) = b, for all , $0 \leq t \leq 1$.

The lifting $\widetilde{C_b}$ has initial point e and satisfies p $\widetilde{C_b} = C_b$

But the path $C_{\tilde{b}}$ defined by $C_{\tilde{b}}$ (t) = e for all t has the initial point e and satisfies

p$C_{\tilde{b}}$ (t) = p e = b = C_b (t), ∀ t

So p $C_{\tilde{b}} = C_b$

Hence $\widetilde{C_b} = C_{\tilde{b}}$

So, e.1 = e. $[C_b]$

= $\widetilde{C_b}$ (1)

= $C_{\tilde{b}}$ (1)

So the correspondence from $p-1(b) \times \pi(B, b) \rightarrow \pi(B, b)$ satisfies the following conditions.
(1) (e. [α]). [β] = e. ([α]. [β])
(2) e.1 = e, for [α], [β] in π (B, b) and $e \in p^{-1}(b)$

Here 1 = $[C_b]$ is the identity element of the group π (B, b).

Thus the mapping from $p-1(b) \times \pi(B, b) \rightarrow \pi(B, b)$ defines an action of the Fundamental group π (B, b) on the fibre set $p^{-1}(b)$

4.1.3 Definition

Let $e \in p^{-1}(b)$, $[\alpha] \in \pi(B, b)$ and $p^{-1}(b)$ is a right π (B, b) space. Then the **Orbit** of the element e denoted by O_e is defined by $O_e = \{e.[\alpha] / [\alpha] \in \pi(B, b)\}$

4.1.4 Definition

Let $p^{-1}(b)$ is a right π (B, b) space and let $e \in p^{-1}(b)$.

Then the **Stabilizer** of the element e denoted by G_e is defined by

$G_e = \{[\alpha] \in \pi(B, b) / e.[\alpha] = e\}$

But e.$[\alpha] = \tilde{\alpha}$ (1) where $\tilde{\alpha}$ is a path in E with initial point $\tilde{\alpha}$ (0) and p$\tilde{\alpha}$ = α

Thus $G_e = \{[p\,\tilde{\alpha}] \in \pi(B, b) / [\tilde{\alpha}] \in \pi(E, e)\}$

= {p*$[\tilde{\alpha}]$ ∈ π (B, b)/ $[\tilde{\alpha}] \in \pi(E, e)$ and p $\tilde{\alpha} = \alpha$}

= p*π (E, e)

Note:
The Stabilizer G_e of $e \in p^{-1}(b)$ is a subgroup of π (B, b) called **isotropy subgroup** corresponding to the element e.

4.1.5 Theorem

Let (E, p) be a covering space of B and $e_0 \in E$ with $p(e_0) = b_0$. Then the map $p^*: \pi(E, e_0) \rightarrow \pi(B, b_0)$ defined by $p^*: [\alpha] = [p\,\alpha]$ is a monomorphism.

4.1.6 Theorem

$p^{-1}(b)$ is a homogenous right π (B, b) space.

Proof :
By using the transitive action property of Abstract Algebra, the reader can easily prove the theorem

4.1.7 Theorem:

For any point e $\in p^{-1}(b)$, the isotropy subgroup corresponding to e is precisely the subgroup $p^*\,\pi$ (E, e) of π (B, b).

Proof:-
By definition, the isotropy subgroup corresponding to e is the set {[α] π (B, b)/ e. [α] = e}

So the isotropy subgroup of e = $\{[p^{\sim}\alpha] \in \pi(B, b) / [^{\sim}\alpha] \in \pi(E, e)\}$

= {p*[$\tilde{\alpha}$] ∈ π (B, b)/ $[\tilde{\alpha}] \in \pi(E, e)$ and p $\tilde{\alpha} = \alpha$}
= p*π (E, e)

Hence the proof.

4.1.8 Theorem

$p^{-1}(b) \simeq \pi(B, b) / p*\pi(E, e)$

Proof:
Proof follows from Theorem 2.1.6 and Theorem 2.1.7 and using the corresponding lemma from [9].

4.1.9 Lemma

Let (E, p) be a covering space of B and $b_0 \in B$. Then the subgroups $p^*\,\pi$ (E, e) for $e \in p^{-1}(b)$ are exactly the conjugacy class of subgroups of π (B, b_0).

4.2 Normal Covering Spaces

In this section we shall discussed about the properties of a normal covering space.

4.2.1 Definition
Let (E, p) be a covering space of B. Then (E, p) is called a **normal covering space**, if there exists a point b ∈ B such that for some $e \in p^{-1}(b)$, the subgroup p*π (E, e) is a normal subgroup of π (B, b).

Eg: The covering space (R, p) of S^1 is a normal covering space of S^1 where $p(t) = e^{2i\pi t}$ for $t \in [0,1]$

4.2.2 Theorem

If (E, p) is a normal covering space of B, then for any b ∈ B and any e ∈ p^{-1} (b), the subgroup p* π (E, e) is a normal subgroup of π (B, b).

Proof

Let (E, p) be a normal covering space of B.

Hence for some b0 ∈B, there is $e_0 \in p^{-1}(b0)$ such that p*π (E, e_0) is normal.

Let $[\alpha] \in \pi(E, e_0)$

For any b ∈ B, and for any $e \in p^{-1}$(b), let γ be a path from e_0 to e

Since E is path connected we have π (E, e0) and π (E, e) are isomorphic.

Let Φ: π (E, e0) →π (E, e) by Φ $[\alpha] = [\gamma^{-1} \alpha \gamma]$ be the isomorphism.

So, Φ π (E, e0) = π (E, e)

Since p* is a monomorphism , we have p* Φ (E, e0) = p* π (E, e).

Since p*π (E, e0) is a normal subgroup of π (B, b0), we have p* π (E, e0) = g^{-1} p* π (E, e0) g, for some $g \in \pi(B, b_0)$

Let $[\eta] \in p*\pi(E, e)$

Then $[\eta]$= p* Φ $[\alpha]$ where $[\alpha] \in \pi(E, e_0)$
= p* $[\gamma^{-1} \alpha \gamma]$
= $[p(\gamma^{-1} \alpha \gamma)]$

That is $[\eta] \in h^{-1}$ p* π (E, e) h where h= $[(p\gamma)^{-1} g (p\gamma)]$ in π (B, b). Thus $p*\pi(E, e) = h^{-1}$ p* π (E, e) h where $h \in \pi(B, b)$

So, $p*\pi(E, e)$ is a normal subgroup of π (B, b).

Hence the proof.

Remark:

We know that if H is a normal subgroup of G, then N [H]= G.

Thus N [p*π (E, e)] = π (B, b)

So, N [p*π (E, e)]/p*π (E, e) = π (B, b)./ p*π (E, e).

4.2.3 Proposition

Let (E, p) be a covering space of B. Let $b \in B$ and $e \in p^{-1}(b)$. If π (B, b) is abelian, then (E, p) is a normal covering space of B

Proof:

Since π (B, b) is abelian, then the subgroup p* π (E, e) is a normal subgroup of π (B, b), It follows that (E, p) is a normal covering space of B.

4.2.4 Theorem

Let (E, p) be a normal covering space of B and $b \in B$. If $e_1, e_2 \in p^{-1}(b)$, then $p*\pi(E, e_1) = p*\pi(E, e_2)$

Proof

Assume that **e1** in E is the point of $p^{-1}(b)$ such that $p*\pi(E, e_1)$ is normal in π (B,b_1). Then, we have $p*\pi(E, e_1)$ and $p*\pi(E, e_2)$ are conjugate subgroups of π (B, b).

That is, we can find an element $g \in \pi(B, b)$ such that $g^{-1} p*\pi(E, e_1) g = p*\pi(E, e_2)$

Since $p*\pi(E, e_1)$ is normal,

$g^{-1} p*\pi(E, e_1) g = p*\pi(E, e_1)$

It therefore follows that $p*\pi(E, e_1) = p*\pi(E, e_2)$

Hence the proof.

5. Conclusion

In this article we have discussed about the normal covering spaces. We have shown that p^{-1} (b) is a homogenius right π (B, b) space. Here we introduce a normal covering space. We also have discussed about the properties of normal covering spaces. For normal covering spaces, the isotropy subgroup is a normal subgroup of π (B, b). This result has many scope for future study. Researchers can correlate this results to other branches of Mathematics. .In this study we have correlated the theories of Abstract Algebra to Algebraic Topology. These concepts can be applied to other branches of Mathematics so as to do collaborative research.

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