Understanding Undefined Quantities: Bridging Mathematical Concepts with Physical and Mathematical Realities

Rabindranath Chattopadhyay

LM (MTA (India), IAPT, IPS, IACS, ISEC, ASI

Abstract: ***This article explores the concept of undefined quantities in mathematics, particularly focusing on the expression (0/0) and similar cases. The discussion delves into how undefined quantities, traditionally confined to theoretical mathematics, find relevance in physical and mathematical realities through examples like Snell's Law and Simple Harmonic Oscillation SHO. By analyzing the interplay between equations, dependent and independent variables, and their practical applications, the article demonstrates how an undefined quantity, often perceived as an abstract concept, can acquire a definitive value or function in specific contexts. This study challenges the conventional perception of undefined numbers and highlights the significance of contextual interpretation in mathematics, emphasizing the transition from purely abstract constructs to application- driven understanding.***

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1. Introduction

Many undergraduate students don't have an idea of an Undefined quantity in the reality absolutely free from confusion. Generally the number (0/0) [zero divided by zero] is called an undefined quantity in Mathematics while there are other expressions too which fall in the same category.

$$\text{Let us write } \frac{0}{0} = \{u_d\} \qquad \text{......(1)}$$

The value of this $\{u_d\}$ may be any number 'x' because the following subsequent equation is valid for all 'x'.

$$0 = 0 \times x \qquad \text{...... (2)}$$

'x' may be 10^{-10} , may be 1..may be (-12), may be (3/7) ..and even may either be '0' or be '∞' ,any number .No definite value of 'x' can be found out from either of the two equations [eqns.(1) & (2)]. That's why this 'x' is called 'undefined value' of $\{u_d\}$or simply 'undefined'. A 'number' simply refers to a symbol representing (2) ordinarily countable value of things or entities. A quantity means an amount that is expressed through a number , number of some kind of unit. But one should keep in mind the inner philosophy of a mathematical equation. The inner philosophy of an equation means the inherent meaning of an equation being the act of equating two quantities, one unknown and the other known and expressed in terms of value. Here a-priori refers to previously known value where a-posteriori means the previously unknown quantity; as if the left hand side represents an operation while the right hand side signify its result, LHS be the cause and RHS be its effect.

Here the eqn.(2) directly follows from eqn.(1) and therefore the starting equation here is eqn.(1). An operation in eqn.(1) is implied in an inverse operation with its sides reversed in eqn.(2) which actually becomes an identity then. If instead eqn.(2) with its sides altered be considered as the starting equation the underlying meaning differs a lot.

$$x \times 0 = 0 \qquad \text{........(3)}$$

To clarify the above paragraph it can be mentioned that where there is an identity as a result of an operation in an equation the preceding form is called the starting equation or simply an equation while the succeeding form is an apparent equation that is actually an identity.

Now this 'x'could have a finite value and obviously then not undefined. This very feature is demonstrated here, one from the point of physical reality and the other from a mathematical reality.

Case(i)(From the point of Physical Reality):[Snell's law is an empirical law in physics, originally established through physical experiments and that is expressed through mathematical equation for some parameters connected to physical variables]

We all know the Snell's law of Refraction of light. It is usually expressed in the following way;

$$_1\mu_2 = \frac{\sin i}{\sin r} \qquad \text{.............(4)}$$

where 'i' is the angle of incidence , 'r' the angle of refraction and $_1\mu_2$ is the refractive index of medium 2 with respect to medium 1. For (3)

Normal incidence $i = r = 0^c$. Then from eqn.(4) one can write

$$_1\mu_2 = \frac{0}{0} = \{u_d\} \qquad \text{.............(5)}$$

This relation is obviously true for normal incidence and emergence of light rays between any pair of optical medium. But from our practical experience we know that for a pair of optical medium and a particular wavelength of light the refractive index is a fixed finite number, for example for glass to air for mean light wavelength yellow 5893Å it is about 1.5 , similarly for water to air it is approximately 1.33 etc. '${}_1\mu_2$' can never be an undefined number. Actually we need here to identify the independent variable, parameter and the dependent variable and rewrite eqn.(5) as the following;

$$sin\, r = \frac{sin\ i}{{}_1\mu_2} \qquad \ldots\ldots\ldots..(6)$$

As ' r ' and so $sin\, r$ is dependent variable one is to obtain the corresponding value of the same for any known value of independent variable 'i' and fixed value of parameter '${}_1\mu_2$'. '${}_1\mu_2$' is obviously not undefined. Prompted from eqn.(6) one gets the value of ' r ' as $r = \sin^{-1}\{\frac{\sin i}{{}_1\mu_2}\} = \sin^{-1}\{\frac{0}{{}_1\mu_2}\} = 0$ [As we know a prior that ${}_1\mu_2 \neq 0$]

Case (ii) (From a point of Mathematical Reality): [SHM or SHO, represented mathematically by the equation that is basically a differential one whose solution is exactly identical with Simple Harmonic Oscillatory function. This SHO-function is certainly a versatile mathematical entity.]

Superposition of two oscillations is simply given by [1]

$$z = A\cos(\omega_1 t + \varphi) + B\cos(\omega_2 t + \psi) =$$
$$= R\ \cos(\mathscr{H} + \eta) \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Where

$$R = \{A^2 + B^2 + 2AB\cos\big((\omega_1 - \omega_2)t + (\varphi - \psi)\big)\}^{\frac{1}{2}} , (4)$$

$$\mathscr{H} = \bar{\omega}\ t + \tan^{-1}[\frac{A\sin\ \varphi + B\sin\ \psi}{A\cos\ \varphi + B\cos\ \psi}] \qquad ..(8)$$

and

$$\eta = tan^{-1}\left[\frac{(A\ 2\ - B\ 2)\ \sin\omega\ t}{A\ \cos\varphi + B\ \cos\psi}\right]$$

where

$$\bar{\omega} = \frac{\omega_1 + \omega_2}{2} \text{ and } \underline{\omega} = \frac{\omega_1 - \omega_2}{2}$$

Now let us consider the well known relation as follows;

$$e^{i\omega t} = \cos\omega t + i\ \sin\omega t \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(9)$$

If one transform eqn.(9) following synthetic transformation[refers to a family of transformation-equations formulated for expressing the result of superposition of SHOs (in general form as oscillatory functions of two associated variables) as another new SHO-function.] [1] like in eqn.(7) then one gets the following results:

$A = 1$, $B = i$, $\omega_1 = -\ \omega_2 = \omega$, $\varphi = 0$, $\psi = {}^{\pi}/_{2}$, $R = (2i\sin 2\omega t)^{\frac{1}{2}}$,

$\mathscr{H} = \tan^{-1} i$, $\eta = \tan^{-1}(\frac{1}{i})$ and so ($\mathscr{H} + \eta$)= $\tan^{-1}(\frac{u_d}{i})$.

It will soon be revealed that this seemingly undefined number (u_d) is actually a defined quantity. Putting these values in eqn.(7) one gets

$$z = (2i\sin 2\omega t)^{\frac{1}{2}} \text{ x } \frac{1}{\sqrt{1 - {u_d}^2}} \ldots\ldots..(10)$$

Now comparing eqns. (7), (9) and (10) one gets $u_d = e^{-2i\omega t}$ and this cannot however be called an undefined quantity.

Putting this value of ' u_d' in eqn.(10) we get obviously

$$z = e^{i\omega t} .$$

In this way it is revealed that apparently undefined number may have in some cases a definite value or be a definite function of variables. (5)

2. Conclusion

Purely mathematical definition of an undefined is completely independent of anything else. But when any such mathematical concept or construct gets entangled into human endeavor through manipulation, incorporation ,computation and simulation inevitably becomes subject to constraints ,conditions and limitations and consequently becomes perception-dependent. In this article this feature has been demonstrated with the help of physical and mathematical realities.

In the analysis of first case the importance of characteristic of the hands of an equation ,on the basis of dependent and independent variables is reviewed along with understanding the meaning of 'undefined' going beyond its ordinary linguistic meaning. And in the second case, on one hand the formula for synthetic transformation of superposed oscillatory function is verified and on the other the usual ordinary perception of 'undefined' as an abstract concept like ($\sqrt{-1}$) , ∞ has been shown to be altered occasionally and having a realistic value or a function . According to its definition a quantity is defined by a standard appropriate amount of some entity depending on the nature of measurable entity. ($\sqrt{-1}$) is neither ordinarily countable nor expresses any material unit and hence being termed as a hypothetical or imaginary number.

References

[1] The Superposition of two S.H.Ms., Rabindranath Chattopadhyay. ***Mathematical Education***, October-December, 1993, pp. 136-139