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Approximation of Sum of Harmonic Progression

Aryan Phadke

Phadke Academy, India
Email: *aryanphadke7[at]gmail.com*

Abstract: ***Background**: The harmonic sequence and the sum of infinite harmonic series are topics of great interest in mathematics. The sum of the infinite harmonic series has been linked to the Euler-Mascheroni constant. It has been demonstrated that, although the sum diverges, it can be expressed as the Euler-Mascheroni constant added to the natural log of infinity. By utilizing the Euler-Maclaurin method, we can extend the expression to approximate the sum of finite harmonic series with a fixed first term and a variable last term. However, natural extension is not possible for a variable value of the first term or the common difference of the reciprocals. **Aim**: The aim of this paper is to create a formula that generates an approximation of the sum of a harmonic progression for a variable first term and common difference. An objective remains that the resultant formula is fundamentally similar to Euler's equation of the constant and the result using the method. **Method**: The principal result of the paper is derived using approximation theory. The assertion that the graph of harmonic progression closely resembles the graph of y=1/x is key. The subsequent results come through a comparative view of Euler's expression and by using numerical manipulations on the Euler-Mascheroni Constant. **Results**: We created a general formula that approximates the sum of harmonic progression with variable components. Its fundamental nature is apparent because we can derive the results of the method from our results.***

Keywords: Approximation theory, Harmonic progression, Euler-Mascheroni constant, Harmonic series

1. Introduction

1.1 Core Concepts

1.1.1 Arithmetic progression

It is the sequence of numbers such that the difference between any two consecutive terms is equal. If the first term of the progression is a with the common difference being d, then the resultant arithmetic progression is as follows
$a, (a+d), (a+2d), (a+3d), \dots\dots\dots, (a+(n-1)d)$

where n is the number of terms.

For the purposes of this paper, we shall use the last term L as the variable instead of the number of terms (n).
$L = a + (n-1)d$

1.1.2 Harmonic progression

It is the sequence of numbers such that each term is a reciprocal of the corresponding term of an arithmetic progression.
The general harmonic progression is represented as follows
$\left(\frac{1}{a}\right), \left(\frac{1}{a+d}\right), \left(\frac{1}{a+2d}\right), \dots\dots\dots\dots\dots, \left(\frac{1}{L}\right)$

1.1.3 Harmonic Series

It is the special case of the sum of harmonic progression where the first term and common difference equal to unity.

$$H(L) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots\dots\dots + \frac{1}{L}$$

To generalize the terms and the terminology, we shall make the following changes.

$$H(L, a, d) = \frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + \dots\dots\dots\dots + \frac{1}{L}$$

1.1.4 Graph of harmonic progression

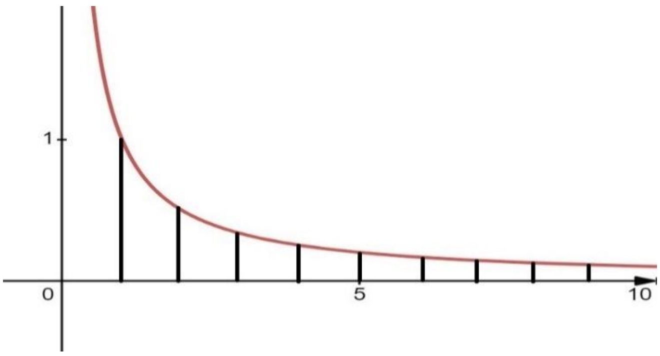

Figure 1: Graph of harmonic progression with a=d=1 superimposed with the function y=1/x

1.2 Background

1.2.1 Euler-Mascheroni constant

It is the limiting difference between the harmonic series and the natural logarithm. It can be represented as follows
$\gamma = \lim_{n\to\infty} [H(n, 1, 1) - \log(n)]$.

The value of the constant can be approximated to 0.5772156649.(N J A Sloane, 2012)

By removing the limit, we get an expression for the infinite harmonic series. (Havil, 2003)

$$\boldsymbol{H(\infty, 1, 1) = \ln(\infty) + \gamma} \qquad (1)$$

1.2.2 Approximation using Euler's constant

With equation (1) and by using the Euler-Maclaurin method, we can arrive at the following expression.

$$H(L, 1, 1) = \ln(L) + \gamma + \frac{1}{2L} - \varepsilon_L$$

where $0 \le \varepsilon_L \le \frac{1}{8L^2}$. (Bressoud, 2007)

As ε_L is negligibly small, especially for sufficiently large values of L, we can ignore it and create an approximation for the partial sum of the harmonic series. The accuracy of which will increase as the value of L increases and as L tends to infinity, we will have arrived at equation (1).

$$H(L,1,1) \approx \ln(L) + \frac{1}{2L} + \gamma \quad (2)$$

1.3 Aims and Objectives

Aim: To find a general formula that approximates the sum of a given harmonic progression.

Objectives:
1) The formula should be applicable for diverse values of a, d and L. This also includes non-integer values.
2) The formula should not require the use of discrete operators such as summation or series expansions. The purpose of this objective is to eliminate the need for any computation.
3) The formula should be fundamentally linked to the Euler's constant and the results of the Euler-Maclaurin method.

2. Results

2.1 General formula

In view of approximation theory, we can assert that the area of the graph of harmonic progression is approximately equal to the area under the curve of $y = \frac{1}{x}$.

By equating the two areas we get an approximation represented as follows

$$\boldsymbol{H(L,a,d) \approx f(L,a,d) = \frac{\ln\left(\frac{L}{a}\right)}{d} + \frac{1}{2a} + \frac{1}{2L}} \quad (3)$$

2.2 Error function and derivation of equation (2)

The next logical step is to introduce an error function that calculates the difference between *H (L, a, d)* and *f(L, a, d)*.

Let
$E(L,a,d) = H(L,a,d) - f(L,a,d)$

The error that can be calculated by this definition is of the infinite harmonic series i.e.
$E(\infty,1,1) = H(\infty,1,1) - f(\infty,1,1)$

We can substitute the expression of *H(L, a, d)* and *f(L, a, d)* when L tends to infinity from equation (1) and equation (3) respectively. By doing so we get

$$\boldsymbol{E(\infty,1,1) = \gamma - \frac{1}{2}} \quad (4)$$

If we assert that $E(L,1,1) \approx E(\infty,1,1)$, we consequently obtain equation (2).

2.3 Formula for a variable first and last term

We have the error of infinite harmonic series from equation (4). To generalize it for a variable first and last term, we can simply introduce another function.
Let
$E(L,a,1) = E(\infty,1,1)\,.\,g\,(L,a,1)$

The absolute value of *g(L, a, 1)* cannot be determined by algebraic manipulation, however we can find an approximation that is accurate and checks certain specific cases for the function.

By doing so, we have the result
$g(L,a,1) \approx \left[\frac{1}{a^2} - \frac{1}{L^2}\right]$

Consequently,
$E(L,a,1) \approx E(\infty,1,1)\,.\left[\frac{1}{a^2} - \frac{1}{L^2}\right]$

$$\boldsymbol{E(L,a,1) \approx \left(\gamma - \frac{1}{2}\right).\left[\frac{1}{a^2} - \frac{1}{L^2}\right]} \quad (5)$$

Thus, we can conclude the formula for the sum of a harmonic progression with variable first and last term as follows
$H(L,a,1) \approx f(L,a,1) + E(\infty,1,1)\,.\,g(L,a,1)$

$$\boldsymbol{H(L,a,1) \approx \ln\left(\frac{L}{a}\right) + \frac{1}{2a} + \frac{1}{2L} + \left(\gamma - \frac{1}{2}\right).\left[\frac{1}{a^2} - \frac{1}{L^2}\right]} \quad (6)$$

2.4 Formula for a variable common difference

We have calculated the error when the first term and the last term are variable in equation (5). Similarly, we can calculate the error when the common difference is variable by introducing another function.
Let
$E(L,a,d) = E(\infty,1,1)\,.\,g(L,a,1)\,.\,k(L,a,d)$

The absolute value of k(L, a, d) largely remains an open problem, but we can approximate it.

The function's dependency on L is very small (<2%) and therefore negligible. This claim can be supported by a comparative analysis of errors at infinity.

The approximation is therefore based on the assertion that
$k(L,a,d) \approx k(a+d,a,d)$

The value of *k(a+d, a, d)* can be calculated manually.
By definition
$$k(a+d,a,d) = \frac{H(a+d,a,d) - f(a+d,a,d)}{E(\infty,1,1).\,g(a+d,a,1)}$$

Because we assume, $k(L,a,d) \approx k(a+d,a,d)$, we can conclude that
$$k(L,a,d) \approx \frac{H(a+d,a,d) - f(a+d,a,d)}{E(a+d,a,1)}$$

Substituting the expressions of these terms from equation (0), equation (3) and equation (5), we get the following formula

$$\boldsymbol{k(L,a,d)} \approx \frac{\left[\frac{1}{2a}+\frac{1}{2(a+d)}-\frac{\ln\left(\frac{a+d}{a}\right)}{d}\right]}{\left(\gamma-\frac{1}{2}\right).\left(\frac{1}{a^2}-\frac{1}{(a+d)^2}\right)} \qquad (7)$$

Thus, we can conclude the formula for sum of harmonic progression with variable first term, common difference and last term is as follows

$$H(L,a,d) \approx f(L,a,d) + \frac{g(L,a,1)}{g(a+d,a,1)} \times E(a+d,a,d)$$

$$\boldsymbol{H(L,a,d)} \approx \frac{\ln\left(\frac{L}{a}\right)}{d}+\frac{1}{2a}+\frac{1}{2L} + \frac{(L^2-a^2).(a+d)^2}{L^2.d.(2a+d)} \times \left[\frac{1}{2a}+\frac{1}{2(a+d)}-\frac{\ln\left(\frac{a+d}{a}\right)}{d}\right] \qquad (8)$$

3. Discussion

3.1 General Formula

The method to obtain equation (3) is based in approximation theory. The principal assumption is that the area of the graph of harmonic progression is approximately equal to the area under the curve of $y = \frac{1}{x}$.

This assertion is valid because
1) Harmonic progression is a partition of the domain of the function.
2) The graphs become similar as the value of *d* decreases.

The area of the graph of harmonic progression [*Ar(HP)*] can be calculated by expressions of areas of simple geometric shapes when Figure (1) is altered in the following way.

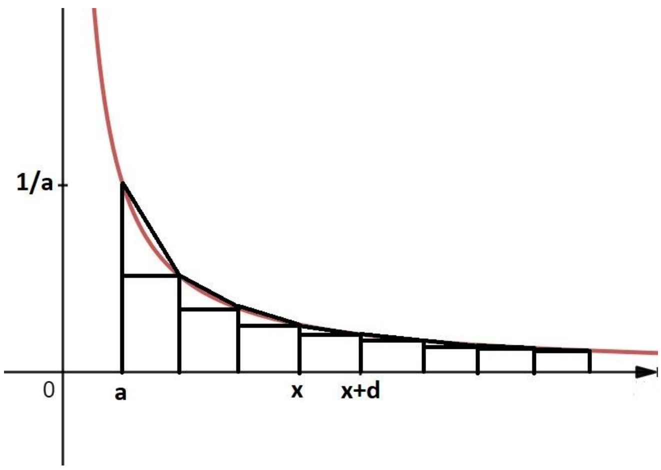

Figure 2: Graph of area of a harmonic progression with a variable term x

The area then becomes a summation of the areas of variable rectangles and triangles.

Area of any rectangle for a variable term x can be represented as $d \times \frac{1}{x+d}$. The sum of the areas of these rectangles will therefore be

$$\sum_{x=a}^{L-d} \frac{d}{x+d} = d \times H(L,a,d) - \frac{d}{a}$$

Similarly, the area of any triangle for a variable term x can be represented as $\frac{1}{2} \times d \times \left(\frac{1}{x}-\frac{1}{x+d}\right)$. The sum of the areas of these triangles will therefore be

$$\sum_{x=a}^{L-d} \frac{d}{2} \times \left(\frac{1}{x}-\frac{1}{x+d}\right) = \frac{d}{2a}-\frac{d}{2L}$$

The total area will be the sum of these two i.e.

$$Ar\ (HP) = d \times \left[H(L,a,d) - \frac{1}{2a}-\frac{1}{2L}\right]$$

The area under the curve is simply the integral of the function i.e. $Ar\left(\frac{1}{x}\right) = \int_a^L \frac{1}{x}dx$

Therefore, the area under the curve can be written as

$$Ar\left(\frac{1}{x}\right) = \ln\left(\frac{L}{a}\right)$$

In line with our assumption $Ar\ (HP) \approx Ar\left(\frac{1}{x}\right)$

$$d \times \left(H(L,a,d) - \frac{1}{2a}-\frac{1}{2L}\right) \approx \ln\left(\frac{L}{a}\right)$$

Hence, we can conclude equation (3)

$$H(L,a,d) \approx f(L,a,d) = \frac{\ln\left(\frac{L}{a}\right)}{d}+\frac{1}{2a}+\frac{1}{2L}$$

3.2 Error function and derivation of Equation (2)

We begin by introducing an error function that is equal to the difference between the sum and the approximation of the sum of a given harmonic progression. i.e.

$$E(L,a,d) = H(L,a,d) - f(L,a,d)$$

Next, we determine the expression for the error of the infinite harmonic series i.e.

$$E(\infty,1,1) = H(\infty,1,1) - f(\infty,1,1)$$

We can substitute these values from equation (1) and equation (3) and we will get

$$E(\infty,1,1) = [\ln(\infty) + \gamma] - \left[\ln(\infty)+\frac{1}{2}+\frac{1}{2\infty}\right]$$

The equation (4) can be obtained by solving this equation.

To derive equation (2), we must assume that the error in a partial sum is approximately equal to the error of the infinite harmonic series. i.e. $E(L,1,1) \approx E(\infty,1,1)$

Consequently, the expression for the partial sum becomes

$$H(L,1,1) \approx f(L,1,1) + E(\infty,1,1)$$

We can substitute of these terms from equation (3) and equation (4). It is apparent without any further manipulation that this results in equation (2)

$$H(L,1,1) \approx \left[\ln\left(\frac{L}{1}\right)+\frac{1}{2}+\frac{1}{2L}\right]+\left[\gamma-\frac{1}{2}\right]$$

3.3 Formula for a variable first and last term

We have obtained the error in the infinite harmonic series. To extend it for a variable first and last term, we can introduce a function that acts a coefficient.

$$E(L,a,1) = E(\infty,1,1) \times g(L,a,1)$$

The absolute value of *g(L, a, 1)* cannot be determined. However, an approximation can be obtained by finding a function that follows certain necessary constraints. We shall now list the constraints and the reasons for including them.

3.3.1 Special case constraints

In the special case of the infinite harmonic series where L tends to infinity, $E(L,1,1) = E(\infty,1,1)$. Therefore,

$$g(\infty,1,1) = 1$$

In the special case where the first term is equal to the last term which is equal to unity. $E(1,1,1) = 0$. Therefore,

$$g(1,1,1) = 0$$

3.3.2 The sum constraint

Consider the sum of error functions of the following harmonic progressions.

$$E(a,1,1) + E(L,a,1) + E(\infty,L,1)$$

We know that

$$E(x,y,z) = H(x,y,z) - f(x,y,z)$$

We know from equation (0) that

$$H(a,1,1)+H(L,a,1)+H(\infty,1,1) = H(\infty,1,1) + \frac{1}{a}+\frac{1}{L}$$

From equation (3) we get

$$f(a,1,1) = \ln\left(\frac{a}{1}\right) + \frac{1}{2}+\frac{1}{2a}$$

$$f(L,a,1) = \ln\left(\frac{L}{a}\right) + \frac{1}{2a}+\frac{1}{2L}$$

$$f(\infty,1,1) = \ln\left(\frac{\infty}{L}\right) + \frac{1}{2L}+\frac{1}{2\infty}$$

Therefore, we have

$$f(a,1,1)+f(L,a,1)+f(\infty,L,1) = \ln(\infty)+\frac{1}{2}+\frac{1}{a}+\frac{1}{L}$$

Thus,

$$E(a,1,1)+E(L,a,1)+E(\infty,1,1) = \gamma-\frac{1}{2} = E(\infty,1,1)$$

Because g is simply the coefficient of E, the sum of the coefficients in this case must be one.

$$g(a,1,1) + g(L,a,1) + g(\infty,L,1) = 1$$

3.3.3 Case for *g(L, a, 1)*

The function that best approximates *g(L, a, 1)* that also follows the constraints listed above was found to be

$$g(L,a,1) \approx \left[\frac{1}{a^2}-\frac{1}{L^2}\right]$$

We shall now verify that the constraints are followed.

$$g(\infty,1,1) = \left[\frac{1}{1^2}-\frac{1}{\infty^2}\right] = 1$$

$$g(1,1,1) = \left[\frac{1}{1^2}-\frac{1}{1^2}\right] = 0$$

For a partial sum of harmonic series where the first term is one and the last term is L, $g(L,1,1) = \left[\frac{1}{1}-\frac{1}{L^2}\right]$.

The approximation of the sum will be

$$H(L,1,1) \approx \ln(L)+\frac{1}{2}+\frac{1}{2L}+\left(\gamma-\frac{1}{2}\right)\times\left[\frac{1}{1}-\frac{1}{L^2}\right]$$

By comparing this with the expression obtained by the Euler-Mclaurin method, we get

$$\varepsilon_L = \frac{\left(\gamma-\frac{1}{2}\right)}{L^2} \approx \frac{1}{13L^2}$$

Which incidentally is within the limits $0 \le \varepsilon_L \le \frac{1}{8L^2}$.

The final constraint is that the sum of the coefficients must equal to unity.

$$\left[1-\frac{1}{a^2}\right]+\left[\frac{1}{a^2}-\frac{1}{L^2}\right]+\left[\frac{1}{L^2}-\frac{1}{\infty^2}\right] = 1$$

Additionally, this approximation is intuitive in that it is simply the difference derivative of $\frac{1}{x}$.

This corresponds to the difference between the slopes of the function in question at the first and the last term.

With all this in mind, we can conclude that our hypothesis is viable and hence equation (5) and equation (6) are valid.

3.4 Formula for a variable common difference

We have obtained the error for a variable first and last terms. To extend it further for a variable common difference, we must introduce another function that acts as a coefficient.

$$E(L,a,d) = E(L,a,1) \times k(L,a,d)$$

Naturally, the absolute value of k(L, a, d) cannot be determined. Additionally, the method of finding a function that closely approximates it that also follows certain constraints does not lend any fruitful results.

The best method to find an approximation for *k(L, a, d)* is to simply calculate a small portion of it manually. Experimentally we saw that *k(L, a, d)* is dependent on *a*, *d*, and *L* as expected. However, its dependency on *L* is negligible (<2%).

If this holds true, and we eliminate L as a variable, we can assert that

$$k(L,a,d) \approx k(a+d,a,d).$$

The latter term can be calculated manually by its definition.

$$\boldsymbol{k(a+d,a,d)} = \frac{\boldsymbol{H(a+d,a,d)} - \boldsymbol{f(a+d,a,d)}}{\boldsymbol{E(a+d,a,d)}} \quad (9)$$

3.4.1 The dependency on *L*

I shall make the case here that although *k(L, a, d)* depends on *L*, it is does so, negligibly.

Because *k(L, a, d)* is directly proportional to *L*, it will have the largest effect on it when *L* is equal to infinity.

Consider the infinite harmonic progression such that $a = d = x$. It is apparent that all its components are equal to the corresponding components of the harmonic series divided by *x*.

$$E(\infty, x, x) = \frac{E(\infty, 1, 1)}{x}$$

We know by definition that

$$k(\infty, x, x) = \frac{E(\infty, x, x)}{E(\infty, 1, 1) \times g(\infty, x, x)}$$

Therefore,

$$k(\infty, x, x) = \frac{\frac{E(\infty,1,1)}{x}}{\frac{E(\infty,1,1)}{x^2}} = x$$

L will have the smallest effect on *k(L, a, d)* when *L* is equal to (*a*+*d*).
Consider the harmonic progression such that $a = d = x$and $L = a + d = 2x$.
Using equation (9), we have

$$k(2x, x, x) = \frac{\left[\frac{1}{x} + \frac{1}{2x}\right] - \left[\frac{\ln\left(\frac{2x}{x}\right)}{2} - \frac{1}{2x} - \frac{1}{4x}\right]}{\left(\gamma - \frac{1}{2}\right) \times \left[\frac{1}{x^2} - \frac{1}{(2x)^2}\right]}$$

Therefore,

$$k(2x, x, x) = \frac{x\left(\frac{3}{4} - \ln(2)\right)}{\frac{3}{4}\left(\gamma - \frac{1}{2}\right)} \approx 0.9817x$$

The difference between the greatest and smallest effect of *L* on *k(L, a, d)* is $0.0182x$. In percent of $k(\infty, x, x) = x$, it would be equal to 1.82% which is less that 2 percent.

3.4.2 The Final general formula

Thus, we can conclude that the effects of *L* on *k(L, a, d)* can be ignored, and we can proceed with equation (9).

$$k(L,a,d) \approx k(a+d,a,d) = \frac{\frac{1}{2a} + \frac{1}{2(a+d)} - \frac{\ln\left(\frac{a+d}{a}\right)}{d}}{\left(\gamma - \frac{1}{2}\right) \times \left[\frac{1}{a^2} - \frac{1}{(a+d)^2}\right]}$$

By definition

$$E(L,a,d) \approx E(\infty, 1, 1) \times g(L, a, 1) \times k(a+d, a, d)$$

$$E(L,a,d) \approx \left[\frac{(L^2 - a^2) \times (a+d)^2}{L^2 d \times (2a+d)}\right] \times \left[\frac{1}{2a} + \frac{1}{2(a+d)} - \frac{\ln\left(\frac{a+d}{a}\right)}{d}\right]$$

Thus, we can conclude equation (8) as the general formula for the approximation of the sum of harmonic progression with variable first, common difference and last term.

3.5 Verification

To verify the approximation and test its accuracy, we shall use five sample harmonic progressions of varying first term and common difference.

For each of them, we will graph the sum and the approximation and calculate the absolute error and expected accuracy.

3.5.1 Case I: *a=d=1*

Consider the harmonic progression such that $a = 1$; $d = 1$.
We will use equation (6) as the approximation.

The absolute error and accuracy can be defined in the following way.

$$E = H(L, 1, 1) - f(L, 1, 1) - E(\infty, 1, 1) \times g(L, 1, 1)$$

$$A = \frac{H(L, 1, 1)}{f(L, 1, 1) + E(\infty, 1, 1) \times g(L, 1, 1)} \times 100$$

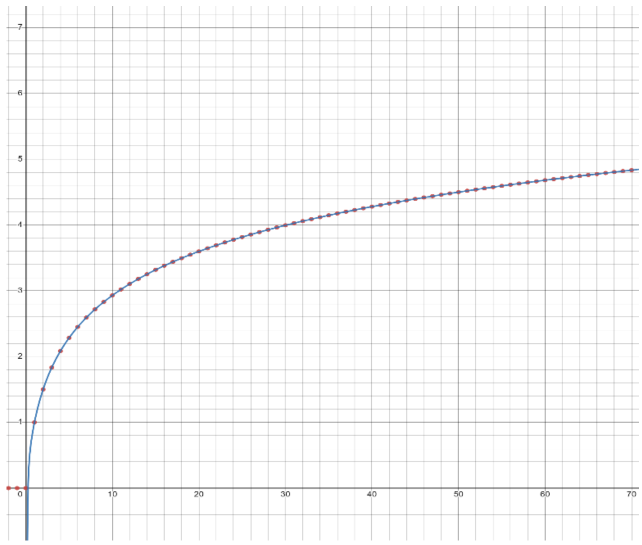

Figure 3: Graph of H(L, 1, 1) (discrete function) superimposed with equation (6) (Continuous function)

The maximum value of absolute error is found to be when $L = 2$; $E(max) = -0.0011$
The minimum accuracy also found at $L = 2$; $Accuracy\ (min) = 99.929\%$
As the value of *L* increases, so does the accuracy and when $L > 12$; $Accuracy > 99.999\%$

3.5.2 Case II: *a>1, d=1*

Consider the harmonic progression such that $a = 75$; $d = 1$
We will use equation (6) as the approximation.

The absolute error and accuracy can be defined in the following way.

$$E = H(L,75,1) - f(L,75,1) - E(\infty,1,1) \times g(L,75,1)$$

$$A = \frac{H(L,75,1)}{f(L,75,1) + E(\infty,1,1) \times g(L,75,1)} \times 100$$

Figure 4: Graph of H(L, 75, 1) (discrete function) superimposed with equation (6) (continuous function)

The maximum value of absolute error and the minimum accuracy both occur when L tends to infinity. Therefore, it cannot be determined but for a rough idea, we will calculate their values for the first 10000 terms.

$$L = 10074\,;E = 1.086 \times 10^{-6}$$

$$L = 10074\,;\ A = 99.9998\%$$

3.5.3 Case III: *a=1. d>1*

Consider the harmonic progression such that $a = 1\,;\ d = 75$

We will use equation (8) as the approximation.

The absolute error and accuracy can be defined in the following way.

$$E = H(L,1,75) - f(L,1,75) - E(\infty,1,1) \times g(L,1,1) \times k(L,1,75)$$

$$A = \frac{H(L,1,75) \times 100}{f(L,1,75) + E(\infty,1,1) \times g(L,1,1) \times k(L,1,75)}$$

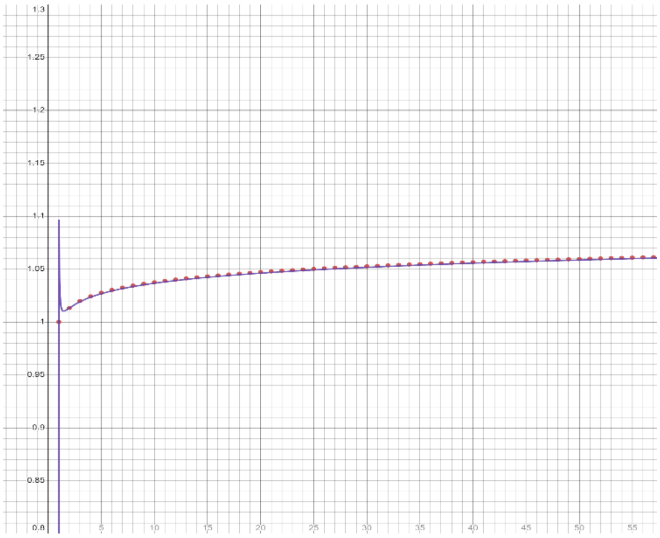

Figure 5: Graph of H(L, 1, 75) (discrete function) superimposed with equation (8) (continuous function)

The maximum value of absolute error and the minimum accuracy both occur when L tends to infinity. Therefore, it cannot be determined but for a rough idea, we will calculate their values for the first 10000 terms.

$$L = 749926\,;\ E = 9.27 \times 10^{-4}$$

$$L = 749926\,;\ A = 99.918\%$$

3.5.4 Case IV: *a>1, d>1*

Consider the harmonic progression such that

$$a = 100\,;\ d = 10$$

We will use equation (8) as the approximation.

The absolute error and accuracy can be defined in the following way.

$$E = H(L,100,10) - f(L,100,10) - E(\infty,1,1).\,g(L,100,1).\,k(L,100,10)$$

$$A = \frac{H(L,100,10) \times 100}{f(L,100,10) + E(\infty,1,1).\,g(L,100,1).\,k(L,100,10)}$$

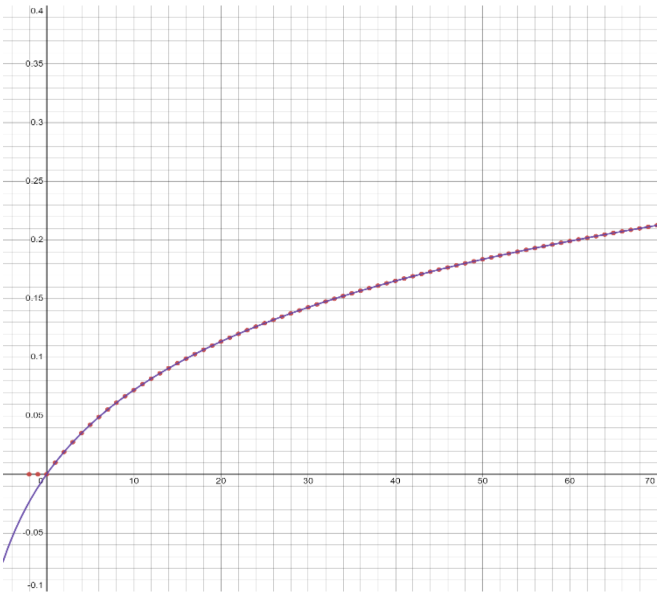

Figure 6: Graph of H(L, 100, 10) (discrete function) superimposed with equation (8) (continuous function)

The maximum value of absolute error and the minimum accuracy both occur when L tends to infinity. Therefore, it cannot be determined but for a rough idea, we will calculate their values for the first 10000 terms.

$L = 100090\,;\ E = 6.828 \times 10^{-8}$

$L = 100090\,; A = 99.9998\%$

3.5.5 Case V: *a, d* are non-integers

Consider the harmonic progression such that $a = \frac{15}{2}\,;\ d = \frac{1}{4}$

We will use equation (8) as the approximation.

The absolute error and accuracy can be defined in the following way.

$$E = H\left(L, \frac{15}{2}, \frac{1}{4}\right) - f\left(L, \frac{15}{2}, \frac{1}{4}\right) - E(\infty, 1, 1).\, g\left(L, \frac{15}{2}, 1\right).\, k\left(L, \frac{15}{2}, \frac{1}{4}\right)$$

$$A = \frac{H\left(L, \frac{15}{2}, \frac{1}{4}\right) \times 100}{f\left(L, \frac{15}{2}, \frac{1}{4}\right) + E(\infty, 1, 1).\, g\left(L, \frac{15}{2}, 1\right).\, k\left(L, \frac{15}{2}, \frac{1}{4}\right)}$$

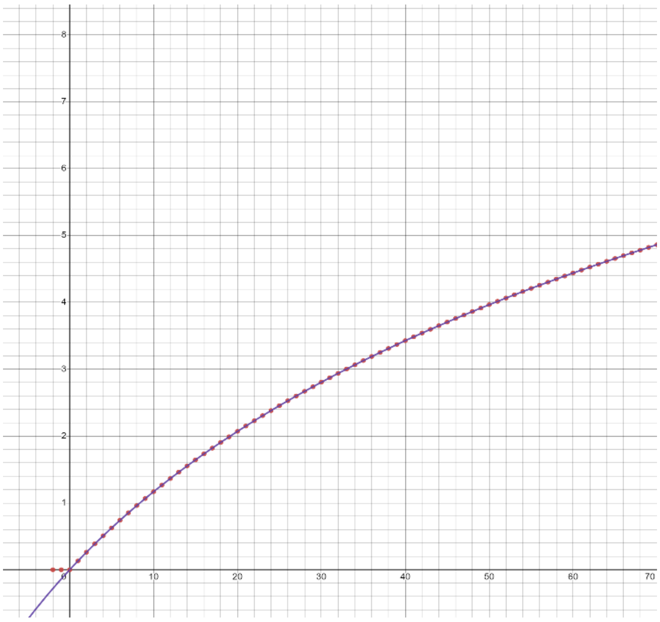

Figure 7: Graph of H (L, 15/2, ¼) (discrete function) superimposed with equation (8) (continuous function)

The maximum value of absolute error and the minimum accuracy both occur when L tends to infinity. Therefore, it cannot be determined but for a rough idea, we will calculate their values for the first 10000 terms.

$L = \frac{10029}{4}\,; E = 3.85 \times 10^{-8}$

$L = \frac{10029}{4};\ A = 99.999999\%$

4. Conclusion

In keeping with the principal aim of the paper, we were able to create a general formula to approximate the sum of a given harmonic progression. Majority of the contemporary approximations are only applicable for the special case of the harmonic series. The result formula is applicable for diverse values of the first term and the common difference which also include non-integer values.

One of the objectives of this paper was to construct a formula that does not depend on discrete operators such as summation. It is apparent that none of the resultant formulas depend on the use of discrete operators or series expansions.

The fundamental nature of the general form must be stressed. It shares a strong connection with the original equation of Euler-Mascheroni constant and is also able to derive the results found by the Euler-Maclaurin method for the harmonic series.

The statistical analysis also reveals positive results. Although the accuracy of the formula will decrease as the last term increases, it will do so very slowly and is highly unlikely to drop below 99.9% regardless of the first term and common difference. If on the other hand, only the absolute error is relevant, then the results are more promising.

4.1 Application

To calculate the sum of a harmonic progression when the common difference is unity, equation (6) should be preferred. For all other cases equation (8) should be used.

The general application of the formula is in any area that requires the sum of a given harmonic progression, such that computation is not a viable path or when the solution is required to be algebraic in nature.

4.2 Future Research

The problem that still requires further attention is perhaps finding a better approximation for *k(L, a, d)* when *d* is very large. A constraint based search for *k(L, a, d)* can also be a worthwhile avenue.

The sum of other series in which the degree of the terms is less than zero such as the finite Basel problem could also potentially be approximated using similar methods.

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