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# Exploring the Uniqueness of Best Simultaneous Approximations in Finite Dimensional Subspaces

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**Abstract:** ***This study explores the uniqueness of best simultaneous approximation of two continuous functions on a closed interval from a finite dimensional subspace. The uniqueness condition is demonstrated to imply that the subspace is Chebyshev. The research examines special case of even and odd function approximations, providing valuable insights into the approximation behavior from finite-dimensional spaces.***

**Keywords:** Simultaneous Approximation, Chebyshev Subspace, Uniqueness, even functions, odd functions, Finite Dimensional Subspaces.

## 1. Introduction

Let $C\,[a\,,b]$ be the set of all real-valued continuous functions defined on the closed interval $[a\,,b]$ with the uniform norm $\|\,.\,\|$.

For $f \in C[a\,,b]$,
$\|f\| = \max\,\{\,|f(x)|, x \in [a\,,b]\}$.

The norms $\|F\|_{A(\infty)}$ on $E = C[a\,,b] \times C[a\,,b]$ is defined as follows:
For $F = (F_1\,,F_2\,) \in E$

$$\|F\|_{A(\infty)} = Max\,\{\,\|F_1\|\,,\|F_2\|$$

Now if $S$ is an n-dimensional subspace of $C[a\,,b]$, then $U = \{\,(s\,,s): s \in S\,\}$ forms an n-dimensional subspace of $E$ and there exist $u^* = (s^*\,,s^*)$ and $v^* = (t^*\,,t^*)$ where $s^*\,,t^* \in S$ such that :

$$\|F - u^*\|_{A(\infty)} = \operatorname*{Inf}_{u\,\in U} \|F - u\|_{A(\infty)}$$
$$= \operatorname*{Inf}_{s\,\in S} Max\,\{\,\|F_1 - s\|\,,\|F_2 - s\|\,\}$$
$$= \|F_k - s^*\| \qquad k = 1\ or\ 2.$$

Such $s^*$ is termed as the best $A(\infty)$ simultaneous approximation to $F = (F_1\,,F_2\,)$ from $S$. The set of all best $A(\infty)$ simultaneous approximations to $F$ from $S$ will be denoted by $P_S(F,\infty)$. And $P_S(F_k)$ is the set of all best uniform approximation to $F_k$ from $S$, $k \in \{\,1\,,2\,\}$.

We are interested in the uniqueness of the best $A(\infty)$ simultaneous approximation of two continuous functions from a finite dimensional subspace. In section two we will show that the uniqueness of the best $A(\infty)$ simultaneous approximation of two continuous functions from a finite dimensional subspace$S$ implies that $S$ is a Chebyshev subspace.

**Definition 1:** A point $t \in [a\,,b]$ is called a straddle point for two functions $f$ and $g$ in $C[a\,,b]$ if there exists $\sigma = \pm 1$ such that

$$\|f\| = \sigma f(t), \;\; \|g\| = -\sigma g(t).$$

**Definition 2:** The functions $f$ and $g \in C[a\,,b]$ are said to have $d$ alternations on $[a\,,b]$ if there exists $d+1$ distinct points $x_1 < \cdots < x_{d+1}$ in $[a\,,b]$ such that for some $\sigma = \pm 1$,

$$f(x_i) = \sigma\|f\|, \;\; if\ i\ is\ odd$$
$$g(x_i) = -\sigma\|g\|, \;\; if\ i\ is\ even$$

or

$$g(x_i) = \sigma\|g\|, \;\; if\ i\ is\ odd$$
$$f(x_i) = -\sigma\|f\|, \;\; if\ i\ is\ even.$$

**Definition 3** Let $S$ be the span of $\{\,u_1, u_2, \ldots, u_n\,\}$ where $u_1, u_2, \ldots, u_n$ are continuous functions on the closed interval $[\,a, b\,]$, we say that $S$ is a Chebyshev subspace of $C[a\,,b\,]$ if

$$Det\,\left[u_i(x_j)\right]_{i,j=1}^{n} \neq 0 \quad \text{for any set of points}$$
$$a \leq x_1 < x_2 < \ldots < x_n \leq b.$$

The notations in this section will be used throughout this paper. Theorems 1 of this section and the remark thereafter which are needed for our analysis, are direct consequences of theorems 1 and 3 of [2].

**Theorem 1** Let $S$ be an n-dimensional Haar subspace of $C[a,b]$ , if $F_1 \geq F_2$ on $[a,b]$ then $s^* \in P_S(F,\infty)$ if and only if $F_1 - s^*$ & $F_2 - s^*$ have a straddle point or $n$ alternations on $[a,b]$ with $\|F_1 - s^*\| = \|F_2 - s^*\|$. Furthermore, if $F_1 - s^*$ & $F_1 - s^*$ have $n$ alternations on $[a,b]$ then $s^*$ is unique.

**Remark**: If $t \in [a,b]$ is a straddle point for $F_1 - s^*$ & $F_2 - s^*$, $F_1 \geq F_2$ on $[a,b]$ then

$$(F_1 - F_2)(t) = (F_1 - s^*)(t) + (F_2 - s^*)(t)$$
$$= \|F_1 - s^*\| + \|F_2 - s^*\| \geq \|F_1 - F_2\|.$$

This implies that $(F_1 - F_2)(t) = \|F_1 - F_2\|$ and $\|F_1 - s^*\| + \|F_2 - s^*\| = \|F_1 - F_2\| \leq \|F_1 - s\| + \|F_2 - s\| \quad \forall s \in S$.

## 2. The Main Result

**Theorem 2** Let $S$ be n-dimensional subspace of $C[a\,,b]$.If the best $A(\infty)$ simultaneous approximation to any two continuous functions on $[a\,,b]$ is unique, then $S$ is a Chebyshev subspace.

**Proof**

Suppose that the conditions of the theorem are satisfied but $S$ is not a Chebyshev subspace. If $S =$ span $\{ u_1, u_2, \ldots, u_n \}$ where $u_1, u_2, \ldots, u_n$ are continuous functions on the closed interval $[a, b]$, then there exists a set of points $a \le x_1 < x_2 < \ldots < x_n \le b$ such that the matrix

$$\begin{bmatrix} u_1(x_1) & u_1(x_2) & \ldots & u_1(x_n) \\ . & . & \ldots & \\ . & . & \ldots & \\ . & . & \ldots & \\ u_n(x_1) & u_n(x_2) & \ldots & u_n(x_n) \end{bmatrix}$$

is singular.

We can select nonzero vectors $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_1, \ldots, b_n)$ to be orthogonal to the columns and rows, respectively, of this matrix. Hence $\sum_{k=1}^{n} a_k u_k(x_i) = 0$ and $\sum_{k=1}^{n} b_k u_j(x_k) = 0$, $i, j = 1, \ldots, n$. Taking $W = \sum_{k=1}^{n} a_k u_k$ then $W(x_i) = 0, i = 1, \ldots n$. And we may assume that $\|W\| < 1$. Select $\{F_1, F_2\} \subseteq C[a, b]$ such that $F_1 \ge F_2$ on $[a, b]$, $\|F_1\| = \|F_2\| = 1$ and $F_1(x_j) = sgn\ b_j$ if $j$ is odd $F_2(x_j) = sgn\ b_j$ if $j$ is even or $F_1(x_j) = sgn\ b_j$ if $j$ is even $F_2(x_j) = sgn\ b_j$ if $j$ is odd. So if $G_1(x) = F_1(x)[1 - |W(x)|]$ and $G_2(x) = F_2(x)[1 - |W(x)|]$, then $G_1(x_j) = F_1(x_j) = sgn\ b_j$ if $j$ is odd and $G_2(x_j) = F_2(x_j) = sgn\ b_j$ if $j$ is even or Then $G_1(x_j) = s\, F_1(x_j) = sgn\ b_j$ if $j$ is odd and $G_2(x_j) = F_2(x_j) = sgn\ b_j$ if $j$ is even or $G_1(x_j) = F_1(x_j) = sgn\ b_j$ if $j$ is even and $G_2(x_j) = F_2(x_j) = sgn\ b_j$ if $j$ is odd.

Now, if $\|G - u\|_{A(\infty)} < 1$ where $G = (G_1, G_2)$ and $u = (s, s)$ for some $s \in S$ then $\|G_1 - s\| < 1$ and $\|G_2 - s\| < 1$ which implies that $sgn\ s(x_j) = sgn\ b_j$, $j = 1, \ldots, n$, contradicting the equation $\sum_{k=1}^{n} b_k\ (x_k) = 0$. Therefore $\|G - u\|_{A(\infty)} \ge 1$ for all $s \in S$.

For $0 \le \lambda \le 1$ we have

$$|G_1(x) - \lambda W(x)| \le |G_1(x)| + \lambda |W(x)| = |F_1(x)|[1 - |W(x)|] + \lambda |W(x)| \le 1 - |W(x)| + \lambda |W(x)| \le 1$$

and

$$|G_2(x) - \lambda W(x)| \le |G_2(x)| + \lambda |W(x)| = |F_2(x)|[1 - |W(x)|] + \lambda |W(x)| \le 1 - |W(x)| + \lambda |W(x)| \le 1.$$

Henceforth $\lambda W$ is a best $A(\infty)$ −simultaneous approximation to $F_1$ and $F_2$ where $0 \le \lambda \le 1$. That is best $A(\infty)$ −simultaneous approximation to $G_1$ and $G_2$ from $S$ is not unique contradicting the assumption and the theorem is proved.

Now we consider different case, taking $S =$ span$\{1, x, x^2, \ldots, x^{n-1}\}$, $[a, b] = [-1, 1]$, $F_1 = x^m$, $F_2 = x^k$ and $F = (F_1, F_2)$ where $n, m$ and $k$ are integers with $k > m \ge n \ge 2$.

**Lemma 1** If $s^* \in P_S(F, \infty)$, then $\|F_1 - s^*\| = \|F_2 - s^*\|$.

**Proof**

First we note that if $m$ is an even integer then $F_1 \ge F_2$ on $[-1, 1]$ and from Lemma (1) of [2] we must have $\|F_1 - s^*\| = \|F_2 - s^*\|$.

If $m$ is an odd integer then $F_1 \ge F_2$ on $[0, 1]$ and $F_2 \ge F_1$ on $[-1, 0]$.

Suppose $\|F_1 - s^*\| > \|F_2 - s^*\|$ then $s^* \in P_S(F_1)$ and there exists a point set $X_1 = \{-1 \le x_1 < x_2 < \ldots < x_{n+1} \le 1\}$ such that $F_1(x_i) - s^*(x_i) = \sigma (-1)^i \|F_1 - s^*\|$, $i = 1, \ldots, n+1$, $\sigma = \pm 1$.

Since $n + 1 > 2$ then there exists a point $y \in X_1 \cap [-1, 0]$ such that $\|F_1 - s^*\| = F_1(y) - s^*(y) \le F_2(y) - s^*(y) \le \|F_2 - s^*\|$ or $t \in X_1 \cap [0, 1]$ such that $\|F_1 - s^*\| = -F_1(t) + s^*(t) \le -F_2(t) + s^*(t) \le \|F_2 - s^*\|$

and this contradict the assumption, hence we must have $\|F_1 - s^*\| = \|F_2 - s^*\|$.

Proceeding as above if $\|F_2 - s^*\| > \|F_1 - s^*\|$ completes the proof.

**Lemma 2**

If both $m$ and $k$ are odd integers and $s^* \in P_s(F, \infty)$ such that $F_1 - s^*$ & $F_2 - s^*$ have $n$ alternations on $[-1, 1]$, then $s^*$ is odd function.

**Proof**

Select $s \in S$ such that $s(x) = -s^*(-x)$ for every $x \in [-1, 1]$, so for any $x \in [-1, 1]$ we have:

$$(F_j - s^*)(x) = F_j(x) + s(-x) = -F_j(-x) + s(-x) \quad j = 1 \text{ or } 2.$$

Which implies that $s \in P_s(F, \infty)$. But $s^*$ is unique, so $s^* = s$ and this completes the proof.

**Theorem 3**

If both $m$ and $k$ are odd integers and $s^* \in P_s(F, \infty)$ such that $F_1 - s^*$ & $F_2 - s^*$ have $n$ alternations on $[-1, 1]$, then $s^* \in P_s(F_j)$ $j = 1$ or $2$.

**Proof**

Let $X = \{-1 \le x_1 < x_2 < \ldots < x_{n+1} \le 1\}$ be the alternating set, that is

$$(F_i - s^*)(x_i) = (-1)^{i+k} \|F_i - s^*\| \quad i = 1, \ldots, n+1, \quad k = 0 \text{ or } 1.$$

$$F_i = \begin{Bmatrix} F_1 & \text{if } i \text{ is odd} \\ F_2 & \text{if } i \text{ is even} \end{Bmatrix}, \text{ or}$$

$$F_i = \begin{Bmatrix} F_2 & \text{if } i \text{ is odd} \\ F_1 & \text{if } i \text{ is even} \end{Bmatrix}$$


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Now assume $k = 0$ and $F_i = \begin{cases} F_1 & if\ i\ is\ odd \\ F_2 & if\ i\ is\ even \end{cases}$,
then:

$$(F_1 - s^*)(y_i) = -\|F_1 - s^*\| \quad i = 1, \dots, q$$
$$(F_2 - s^*)(z_i) = \|F_2 - s^*\| \quad i = 1, \dots, r,$$

where $Y = \{y_1, y_2, \dots, y_q\}$, $Z = \{z_1, z_2, \dots, z_r\}$, $X = Y \cup Z$ and
$n + 1 = q + r$. If $n + 1$ is even then $r = q$ and if $n + 1$ is odd then $r = q + 1$ or
$q = r + 1$.

By lemma 3 $s^*$ is odd function and we have:

$$(F_1 - s^*)(-y_i) = \|F_1 - s^*\| \quad i = 1, \dots, q$$
$$(F_2 - s^*)(-z_i) = -\|F_2 - s^*\| \quad i = 1, \dots, r$$

From the characterization theorem of best uniform approximation, it follows that
$s^* \in P_s(F_j)$ $j = 1, 2$ if $n$ is odd. And if $n$ is even with $q = r + 1$ then
$s^* \in P_s(F_1)$ also $s^* \in P_s(F_2)$ if $r = q + 1$. The proof is the same for the other cases.

## References


[1] Cheney, E.W. Introduction to Approximation Theory. McGraw-Hill, NewYork- London, (1966).

[2] Asiry,M.and Watson, G. A., Simultaneous approximation from Chebyshev and weak Chebyshev spaces, Comm.In Applied Analysis,(2000) **4,** No3.

[3] Asiry, M.and Watson,G. A.,On solution of a class of best Simultaneous approximation problems, International Journal of Computer Mathematics,(1999) **75,** pp 413-425

[4] Alyazidi-Asiry, M., Adjoining a Constant Function to N-Dimensional Chebyshev Space, Journal of Function Spaces, vol. 2016, Article ID 4813979.

[5] Alyazidi-Asiry, M., Extending a Chebyshev Subspace to a weak Chebyshe subspace of higher dimension and Related Results"Journal of Applied and computational Mathematics, **6,** (2017).

[6] Alyazidi, M., On the uniform and simultaneous Approximations of Functions, Journal of Advances of Pure Mathematics, **11**,(2021)

[7] Dunham, C.B. Simultaneous Chebyshev approximation of two functions on an Interval, Proc. A. M. S. **18** (1967), 472-477.

[8] Pinkus, A. Uniqueness in vector- valued approximation, J. Approx. Theory **91** (1997) 17-92.

[9] Watson, G. A. A characterization of best simultaneous approximation, Approx. Theory **75** (1993) 175-182.

[10] Watson. G. A, approximation Theory and Numerical Methods, Jhon Wiley, Chichester, (1980).