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On a Hypersurface of a Conformal β-Change

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Abstract: ***We have considered the conformal β - change of the Finsler metric is given by $\bar{L} = e^{\sigma} f(L, \beta)$, where σ is a function of x , $\beta(x,y) = b_i(x)y^i$ is a 1-form on the underlying Manifold M^n and $f(L, \beta)$ is a homogeneous function of degree one in L and β . In this paper we have studied some properties of hypersurface of a Conformal β - change.***

Keywords: Conformal β - change, Hypersurface, Unit normal vector, Finsler space.

Subject Classification: 53B40, 53C60

1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space on the differentiable manifold M^n equipped with the fundamental function L(x,y), B. N. Prasad, Bindu Kumari and C. Shibata [1], [2] have studied the β-change that is $\bar{L} = f(L, \beta)$, where f is positively homogeneous function of degree one in L and β and β is given by $\beta(x,y) = b_i(x)y^i$ is a 1-form on M^n.

The conformal theory of Finsler space was initiated by M.S.Knebelman [7] in 1929 and has been investigated in detail by many authors C Hashiguchi [8] , Izumi [3] , [4] and Kitayama [10].

The Conformal change is defined as $\bar{L}(x, y) = e^{\sigma(x)}$L (x, y), where σ is a function of position only and known as Conformal factor. The β-change of special Finsler space has been studied by H. S. Shukla, O. P. Pandey and Khageshwar Mandal [7]. In 2018, H.S.Shukla and Neelam Mishra had studied the some properties of conformal β – change [5].

The conformal β – change of the Finsler metric is defined by

$$\bar{L} = e^{\sigma} f(L, \beta) \qquad (1.1)$$

Where, $\beta(x,y) = b_i(x)y^i$ and b_i is 1 – form . We have called this change as conformal β – change of Finsler metric.

In this paper we investigate some properties of hypersurface of a conformal β-change. The Finsler space equipped with the metric $\bar{L}$ given by (1.1) will be denoted by $\bar{F}^n$ .Throughout the paper the quantites corresponding to $\bar{F}^n$ will be denoted by putting bar on the top of them . We shall denote the partial derivatives with respect to x^i *and* y^i *by* ∂_i *and* $\dot{\partial}_i$ respectively.

The homogeneity off gives

$$L f_1 + \beta f_2 = f \qquad (1.2)$$

Where subscripts 1 and 2 denotes the partial derivatives with respect to L and β respectively.

Differentiating (1.2) with respect to L and β respectively, we get

$$L f_{11} + \beta f_{12} = 0 \text{ and } L f_{12} + \beta f_{22} = 0 \quad (1.3)$$

Hence we have

$$\frac{f_{11}}{\beta^2} = -\frac{f_{12}}{L\beta} = \frac{f_{22}}{L^2} \ ,$$

which gives

$$f_{11} = \beta^2 \omega \ , \quad f_{12} = -L\beta\omega \ , \quad f_{22} = L^2 \omega \ (1.4)$$

where Weierstrass function ω is positively homogeneous function of degree - 3 in L and β .

Therefore

$$L\omega_1 + \beta\omega_2 + 3\omega = 0 \quad (1.5)$$

Again ω_1 *and* ω_2 are positively homogeneous function of degree – 4 in L and β , so that

(a) $L\,\omega_{11} + \beta\omega_{12} + 4\omega_1 = 0$, (1.6)
(b) $L\omega_{21} + \beta\omega_{22} + 4\omega_2 = 0$.

Throughout the paper we frequently use equations (1.2) to (1.7) without quoting them.

2. Fundamental quantities of $(M^n, \bar{L})$:

To find the relation between the fundamental quantites of (M^n, L) and $(M^n, \bar{L})$, we use the following results :

$$\dot{\partial}_i \beta = b_i \quad , \dot{\partial}_i L = l_i \quad , \quad \dot{\partial}_j l_i = L^{-1} h_{ij} \quad (2.1)$$

where h_{ij} are components of angular metric tensor of ($M^n, L)$ given by

$$h_{ij} = g_{ij} - l_i l_j = L\dot{\partial}_i \dot{\partial}_j L \ .$$

The successive derivatives of (1.1) with respect to y^i *and* y^j gives ,

$$\bar{l}_i = e^{\sigma}(f_1 + \tau f_2) l_i + e^{\sigma} f_2 m_i \quad (2.2)$$

$$\bar{h}_{ij} = \frac{e^{2\sigma} f f_1}{L} h_{ij} + e^{2\sigma} f L^2 \omega m_i m_j \quad (2.3)$$

where $m_i = b_i - \tau l_i$ and $\tau = \frac{\beta}{L}$.

From (2.2) and (2.3) , we get the following relation between the metric tensors of (M^n, L) and $(M^n, \bar{L})$

$$\bar{g}_{ij} = q_{-1} g_{ij} + q_{-2} l_i l_j + q_{-3}(l_i m_j + l_j m_i) + q_{-4} m_i m_j \qquad (2.4)$$

where
$q_{-1} = \frac{e^{2\sigma}ff_1}{L}$, $q_{-2} = e^{2\sigma}(fL^2\omega + {f_2}^2)\tau^2$, $q_{-3} = e^{2\sigma}f_2(f_1 + \tau f_2)$, $q_{-4} = e^{2\sigma}(fL^2\omega + {f_2}^2)$.

The contravariant components $\bar{g}^{ij}$ of the metric tensor of $(M^n, \bar{L})$ will be derived from (2.4) as follows:

$$g^{-ij} = p_{-1}g^{ij} + p_{-2}l^i l^j + p_{-3}(l^i m^j + m^i l^j) + p_{-4}m^i m^j \quad (2.5)$$

where,

$$p_{-1} = \frac{1}{q_{-1}} ,$$

$$p_{-2} = \frac{L^2}{e^{2\sigma}f^2 f_1 t}\left[pLf^2\left(\frac{f\beta}{L^2} - \Delta f_2\right) - f\omega\beta^2 - 2p\tau\right] ,$$

$$p_{-3} = -\frac{L^2 f_2}{e^{2\sigma}f^2 t} ,$$

$$p_{-4} = -\frac{L^4\omega}{e^{2\sigma}ff_1 t} ,$$

$$l^i = g^{ij}l_j , \quad b^2 = g_{ij}b_i b_j .$$

p = $f_1 f_2 - fL\beta\omega, t = f_1 + L^3\omega\Delta$, $\Delta = b^2 - \tau^2$, $q = 3f_2\omega + f\omega_2$. (2.6)

(a) $\dot{\partial}_i f = \frac{e^{\sigma}f}{L}l_i + e^{\sigma}f_2 m_i$ (2.7)

(b) $\dot{\partial}_i f_1 = -e^{\sigma}L\beta\omega m_i$,

(c) $\dot{\partial}_i f_2 = e^{\sigma}L^2\omega m_i$,

(d) $\dot{\partial}_i \omega = -\frac{3\omega l_i}{L} + \omega_2 m_i$,

(e) $\dot{\partial}_i b^2 = -2C_{..i}$,

(f) $\dot{\partial}_i \Delta = -2C_{..i} - 2\frac{\beta}{L^2}m_i$.

(a) $\dot{\partial}_i p = -L\beta q m_i$ (2.8)

(b) $\dot{\partial}_i t = -2L^3\omega C_{..i} + (L^3\Delta\omega_2 - 3L\beta\omega)m_i$,

(c) $\dot{\partial}_i q = -\frac{3q}{L}l_i + (4f_2\omega_2 + 3L^2\omega^2 + f\omega_{22})m_i$,

Where, denotes the contraction with b^i viz. $C_{..i} = C_{jki}b^j b^k$.

Differentiating (2.4) with respect to y^k and using (2.1) and (2.7), we get

$$\bar{C}_{ijk} = q_{-1}C_{ijk} + U_{ijk} \quad (2.9)$$

$$U_{ijk} = U_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + U_{-2}m_i m_j m_k,$$

where $U_{-1} = \frac{p}{2L}$, $U_{-2} = \frac{qL^2}{2}$

3. Hypersurface of a conformal β-change

The hypersurface $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ of the finsler space $F^n = (M^n, L)$ is given by the equation $x^i = x^i(u^\alpha)$, where α = 1 , 2 , 3 , ………., n-1. The supporting element y^i at a point u = u^α of M^{n-1} is assumed to be tangent to M^{n-1} .

$$y^i = B^i_\alpha(u)v^\alpha \quad (3.1)$$

where $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is the matrix of projection factors of rank n-1 can be assumed as the components of linearly independent vectors that are tangent to F^{n-1}. At every point of u^α of F^{n-1} , a unit normal vector B^i is defined as [9] ,

$$g_{ij}B^i B^j = 1 \text{ and } g_{ij}B^j B^i_\alpha = 0 \quad (3.2)$$

The induced metric tensor $g_{\alpha\beta}$ and induced Cartan tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given as follows [9],

$$\text{(a) } g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta \quad (3.3)$$

and (b) $C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma$

Now we obtain the condition under which the hyperurface of the transformed Finsler space $\bar{F}^n$ to be the normal vector.

Let $\bar{F}^{n-1} = (M^{n-1}, \underline{L}(u,v))$ be a Finslerian hypersurface of the transformed Finsler space $\bar{F}^n$.

The unit normal vector $\bar{B}^i(u,v)$ of $\bar{F}^{n-1}$ is uniquely identified as

$$\bar{g}_{ij}B^i_\alpha \bar{B}^j = 0 \text{ and } \bar{g}_{ij}\bar{B}^i\bar{B}^j = 1 \quad (3.4)$$

The $\bar{B}^\alpha_i$ is the inverse projection factor of $\bar{B}^i_\alpha$, is uniquely defined by

$$\bar{B}^\alpha_i = g_{ij}\bar{g}^{\alpha\beta}B^j_\beta \quad (3.5)$$

where $\bar{g}^{\alpha\beta}$ is the inverse metric tensor of the metric tensor $\bar{g}_{\alpha\beta}$ along $\bar{F}^{n-1}$.

In view of equations (3.4) and (3.5) it follows that

$$\bar{B}^i_\alpha \bar{B}^\beta_i = \delta^\beta_\alpha , \quad \bar{B}^i_\alpha \bar{B}_i = 0 , \bar{B}^i \bar{B}^\alpha_i = 0 , \bar{B}^i \bar{B}_i = 1 \quad (3.6)$$

Transvecting the equation (3.2) by v^α and using $B^i_\alpha v^\alpha = y^i$, we get

$$B_j y^j = 0 \quad (3.7)$$

Contracting the equation (2.4) by $B^i B^j$ and using (3.3) and (3.6) , we obtain

$$\bar{g}_{ij}B^i B^j = q_{-1} + q_{-4}(B^i m_i)^2 \quad (3.8)$$

which demonstrate that $B^i \Big/ \sqrt{q_{-1} + q_{-4}(B^i m_i)^2}$ is a unit normal vector.

Again contracting (2.4) by $B^i_\alpha B^j$ and using (3.3) , (3.6) we get

$$\bar{g}_{ij}B^i_\alpha B^j = (q_{-3}l_i + q_{-4}m_i)B^i_\alpha(B^j m_j) \quad (3.9)$$

Which demonstrates that the vector B^j normal to $\bar{F}^{n-1}$ if and only if

$$(q_{-3}l_i + q_{-4}m_i)B^i_\alpha (B^j m_j) = 0 \quad (3.10)$$

This shows that at least one of the following conditions is correct

$$(a)\ (q_{-3}l_i + q_{-4}m_i)B^i_\alpha = 0 \quad (3.11)$$
$$(b)\ B^j m_j = 0.$$

Transvecting condition (3.11) (a) by v^α gives L=0 which is not possible.

Therefore condition (3.11) (b) holds i.e. ,

$$B^j m_j = 0 \qquad (3.12)$$

In view of (3.7), the equation (3.12) can be equivalently written as

$$B^j b_j = 0 \qquad (3.13)$$

According to the equation (3.8) , (3.9) and (3.13) , we get

$$\bar{B}^i = \frac{B^i}{\sqrt{(q_{-1})}}, \qquad (3.14)$$

which gives

$$\bar{B}_i = \bar{g}_{ij}\bar{B}^j = \sqrt{(q_{-1})}B_i \quad (3.15)$$

Thus, we have

Theorem: If $\bar{F}^{n-1}$ is the hypersurface of the space $\bar{F}^n$ then the vector b_i is tangential to the hypersurface F^{n-1} if and only if each vector normal to F^{n-1} is also normal to $\bar{F}^{n-1}$ **.**

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