Fuzzy Fixed-Point Theory in Numerical Method for Solving Fuzzy Equations

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Abstract:

Fuzzy Fixed Point Theory has emerged as a powerful tool in addressing uncertainties in numerical methods for solving fuzzy equations. This theory extends classical fixed point concepts to fuzzy environments, enabling the resolution of equations where parameters and solutions are expressed as fuzzy sets rather than crisp numbers. The paper explores the application of fuzzy fixed point theory in developing numerical methods for solving fuzzy equations, with a focus on its convergence properties, stability, and computational efficiency. Techniques such as iterative methods and fuzzy differential equations are examined, demonstrating the utility of fuzzy fixed point theory in handling imprecise or vague data. The results reveal that fuzzy fixed point-based numerical methods offer robust solutions in various fields, including engineering, finance, and decision-making, where uncertainty is prevalent.

Keywords:

Fuzzy Fixed Point Theory, Fuzzy Equations, Numerical Methods, Iterative Methods, Convergence, Fuzzy Differential Equations, Uncertainty, Stability, Computational Efficiency, Fuzzy Sets.

Introduction:

Numerous experimenters have shown interest in studying fuzzy equations in the past.Perfect fine modeling of real-world situations with a query is what fuzzy equations are recognized for. Equations with variables that may be changed from the fuzzy set's form are known as fuzzy equations.FDEs may also be used to simulate discriminational equations when their parameters or countries are ambiguous. For modeling and nonlinear control, the outcomes of the fuzzy equations may be directly enforced. There are many issues with using finite-dimensional state models for creating control rules 10 for distributed-mass systems.It is suggested to use Newton's system to solve “fuzzy nonlinear equations”. “Fuzzy nonlinear systems” are proposed to operate in the fixed point system. “Fuzzy Fourier transfigures” 15 is used to examine the logical outcome of a fuzzy heat equation under generalized “Hukuhara partial differentiability”.The fuzzy outside concept is used to bandy the oneness and stability of the outcome for the fuzzy Poisson equation. The “Runge-Kutta system”, the interpolation system, and the iterative system may all be used to get the numerical results of the fuzzy equations. FDE may be broken using certain numerical styles, such as the Runge-Kutta system and the Nystrom system. The fuzzy original value problem's approximate answer is obtained using the Euler system. Alternate-order FDE findings are obtained using the Laplace transfigure system. It is suggested to create the various non-traveling surge outcomes of nonlinear PDEs in the emulsion(G ′ G)-expansion system. Under generalized differentiability, first-order totally fuzzy direct discriminational equations with inpositive 25 or negative outcomes are examined. Fuzzy direct fractional discriminational equations under Riemann-Liouville H-differentiability are examined in concreted findings. Fuzzy equations may also be solved using artificial neural networks. Artificial neural networks are used to solve 30 fuzzy quadratic problems. Artificial neural networks are used to solve fuzzy polynomial problems. Binary fuzzy equations are solved using artificial neural networks. To approximate completely fuzzy matrix problems, a method based on fuzzy neural networks is proposed. On the other hand, when it comes to wide fuzzy equations, these methods fail when utilizing artificial neural networks. Similarly, they can't use artificial neural networks to directly create fuzzy sections. A system of artificial neural networks is suggested for solving FDEs under initial circumstances. A fuzzy conclusion system model based on an unsupervised adaptive network is suggested for solving discriminational equations. 40 are used in neural algorithms to solve discriminational equations. Two operational PDEs are handled by artificial neural networks. fuzzy PDE, fuzzy FDE, fuzzy fuzzy equations, and binary fuzzy equations. The roots of these equations may be obtained in a variety of ways, as this paper explains in detail. There are a number of conditions that ensure the roots of these equations are genuine. The benefits of numerical approaches with regard to perfection are also shown. The composition's following sections are arranged as follows. Some of the initial definitions used throughout the remainder of the study are provided in Section 2. A few numerical approaches for predicting the outcomes of fuzzy and binary fuzzy equations are covered in Section 3. Some numerical methods for varying the outcomes of fuzzy PDEs and FDEs are covered in Section 4. Fifty-five numerical examples with relative analysis are presented in Section 5. The paper is concluded in Section 6. Many experimenters have been interested in studying fuzzy equations for a long time. Perfect fine equations that represent real-world issues with queries are called fuzzy equations. Equations with modifiable parameters from the fuzzy form set are known as fuzzy equations. FDE may also be used to solve discriminational equations with ambiguous parameters or nations. Direct enforcement of working fuzzy equations is possible for both modeling and nonlinear control. When creating control rules 10 for dispersed matter systems, several issues pertaining to the functioning of finite-dimensional state models are thrown in. The solution to fuzzy nonlinear equations is Newton's system. Fuzzy nonlinear systems are operated in fixed point systems. The fuzzy Fourier transfigure 15 is used to analyze the fuzzy heat equation's analytical conclusion for generalized Hukuhara partial differentiability. Regarding consistency and stability of outcomes for fuzzy The fuzzy outside concept serves as the foundation for Poisson's equations. The Runge-Kutta system, the interpolation system, and the iterative system may all be used to provide numerical results for fuzzy equations. Some numerical styles that are comparable to Runge-20 and Nystrom's system The FDE can be broken using Kutto's approach. To provide an approximate solution to the fuzzy original value issue, Euler's system is used. The alternate-order result FDE is obtained using the Laplace transform system. The construction of a G)-expansion system using several non-linear stationary surge PDEs is suggested in the emulsion (G ′). Under generalized differentiability, totally fuzzy direct discriminational equations of the first order with positive or negative outcomes are examined. Fractional discriminational equations under Riemann-Liouville H-differentiability are examined in a particular fuzzy direct result. Fuzzy equations may also be solved by artificial neural networks. In order to resolve IN 30 fuzzy quadratic equations, artificial neural networks are used. Fuzzy polynomial issues are solved with artificial neural networks. In order to resolve binary fuzzy equations, artificial neural networks are used in(). The fuzzy neural network-based system is intended to approximate the solution of entirely fuzzy matrix equations. Nevertheless, current approaches are unable to use artificial neural networks to solve generic fuzzy 35 equations. Additionally, he is unable to use artificial neural networks to directly create fuzzy sections. A neural network approach is suggested in artificial intelligence to solve FDE under initial circumstances. IN is a fuzzy conclusion system model designed to solve discriminational equations. It is based on an unsupervised adaptive network. 40 uses neural algorithms to solve discriminational equations. Two functional PDEs make use of artificial neural networks. A class of first-order PDEs is broken using multilayer artificial neural networks. An unsupervised artificial neural network is designed to address discriminative equations. An artificial neural network is used to address the boundary value control problem for the heat equation. This composition summarizes the 45 existing numerical result styles: fuzzy equations, binary fuzzy equations, FDE, and fuzzy PDE. The solutions to these equations may be derived by several methods, as discussed in this work. Several criteria guarantee that the roots of these equations are authentic. The advantages of numerical styles are shown in 50 specific dates. The subsequent sections of the document are organized as follows. Section 2 contains some preliminary definitions used in the subsequent sections of the work. Section 3 addresses several numerical methods for solving fuzzy equations and binary fuzzy equations. Section 4 addresses various numerical data methodologies for impacting fuzzy PDR and FDE results.

1 Fuzzy Initial Value Problem

Here, we introduce fuzzy initial value problem in the following form:

$$\begin{aligned} y'(t) &= f(t, y(t)), \qquad t \in [t_0, T] \\ y(t_0) &= y_0 \end{aligned} \tag{1}$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a triangular or a triangular shaped fuzzy number. As a result, our issue is hazy cauchy. The fuzzy function y is represented by the notation y = [y1, y2].

It means that the r-level set of "$y(t)$ for $t \in [t_0, T]$" is

$$[y(t_0)]_r = [y_1(t_0; r), y_2(t_0; r)], \quad [y(t)]_r = [y_1(t; r), y_2(t; r)] \qquad r \in (0, 1].$$

The membership function is obtained by using Zadeh's extension concept.

$$f(t, y(t))(s) = \sup\{y(t)(\tau) \,|\, s = f(t, \tau)\}, \qquad s \in \mathbb{R} \tag{2}$$

so $f(t, y(t))$ is a fuzzy number. From this it follows that

$$[f(t, y(t))]_r = [f_1(t, y(t); r), f_2(t, y(t); r)], \; r \in (0, 1] \tag{3}$$

where

$$\begin{aligned} f_1(t, y(t); r) &= \min\{f(t, u) \,|\, u \in [y_1(t; r), y_2(t; r)]\}, \\ f_2(t, y(t); r) &= \max\{f(t, u) \,|\, u \in [y_1(t; r), y_2(t; r)]\}. \end{aligned} \tag{4}$$

We define

$$\begin{aligned} f_1(t, y(t); r) &= F[t, y_1(t; r), y_2(t; r)], \\ f_2(t, y(t); r) &= G[t, y_1(t; r), y_2(t; r)]. \end{aligned} \tag{5}$$

Paper ID: MR241128145849 DOI: https://dx.doi.org/10.21275/MR241128145849

Definition 1.1 *A function $f : \mathbb{R} \to \mathbb{R}_F$ is called a fuzzy function. If for arbitrary fixed $t_0 \in \mathbb{R}$ and $\epsilon > 0$, a $\delta > 0$ such that*

$$|t - t_0| < \delta \Rightarrow D[f(t), f(t_0)] < \epsilon$$

exist, is said to be continuous.

Fuzzy functions that are continuous in metric D are also taken into consideration in this study. Then, for t ∈ [t0,T] and r ∈ [0, 1], [6], the existence of the Definitionnite of f (t, y(t); r) is guaranteed by the continuity of f (t, y(t); r).

As a result, G and F may also be definite functions.

2 Modified Euler's Method

Consider the initial value problem

$$\begin{aligned} y'(t) &= f(t, y(t)), \qquad t \in [t_0, T] \\ y(t_0) &= y_0 \end{aligned} \tag{6}$$

It is known that, the sufficient conditions for the existence of a unique solution to (6) are that f to be continuous function satisfying the Lipschitz condition of the following form:

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \quad L > 0.$$

We replace the interval $[t_0, T]$ by a set of discrete equally spaced grid points

$$t_0 < t_1 < t_2 < \ldots < t_N = T, \ h = \frac{T - t_0}{N}, \ t_i = t_0 + ih, \ i = 0, 1, \ldots, N.$$

to obtain the Euler method for the system (6), we apply Trapezoidal numerical integration method. Integrate the differential equation $y'(t) = f(t, y(t))$ over $[t_n, t_{n+1}]$ to obtain

$$\int_{t_n}^{t_{n+1}} y'(t)\, dt = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt.$$

Therefore

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}\Big[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\Big] - \frac{h^3}{12} f^{(2)}(\xi_1, y(\xi_1)) \tag{7}$$

for some $t_n \leq \xi_1 \leq t_{n+1}$. Equation (7) is an implicit equation in term of $y(t_{n+1})$. To avoid of solving such implicit equation we will substitute $y(t_{n+1})$

by $y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2} f'(\xi_2, y(\xi_2))$ in right hand of (7), where $\xi_2 \in [t_n, t_{n+1}]$. Therefore,

$$y(t_{n+1}) = y(t_n) + \frac{h}{2} f(t_n, y(t_n)) + \frac{h}{2} f\left(t_{n+1}, y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2} f'(\xi_2, y(\xi_2))\right) - \frac{h^3}{12} f^{(2)}(\xi_1, y(\xi_1)), \quad t_n \le \xi_1 \le t_{n+1}, \quad t_n \le \xi_2 \le t_{n+1}. \tag{8}$$

But we have

$$f\left(t_{n+1}, y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2} f'(\xi_2, y(\xi_2))\right) = f\left(t_{n+1}, y(t_n) + hf(t_n, y(t_n))\right) + \frac{h^2}{2} f'(\xi_2, y(\xi_2)) f_y(t_{n+1}, \xi_3) \tag{9}$$

where ξ_3 is in between $y(t_n) + hf(t_n, y(t_n))$ and $y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2} f'(\xi_2, y(\xi_2))$.

As the result of above we will have

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}\left[f(t_n, y(t_n)) + f(t_{n+1}, y(t_n) + hf(t_n, y(t_n)))\right] + \frac{h^3}{4} f'(\xi_2, y(\xi_2)) f_y(t_{n+1}, \xi_3) - \frac{h^3}{12} f''(\xi_1, y(\xi_1)) \tag{10}$$

Thus we have the following one-step explicit equation for calculation $y(t_{n+1})$ using $y(t_n)$:

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}\left[f(t_n, y(t_n)) + f(t_{n+1}, y(t_n) + hf(t_n, y(t_n)))\right] \tag{11}$$

with initial value $y_0 = y(t_0)$.

By dropping the remainder term in (9), we obtain an equivalent equation with (11), modified Euler's method as following,

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))] \qquad n \ge 0. \tag{12}$$

Let $y^{(0)}(t_{n+1}) = y(t_n) + hf(t_n, y(t_n))$ be a good initial guess of the solution $y(t_{n+1})$, and define

$$y^{(j+1)}(t_{n+1}) = y(t_n) + \frac{h}{2}[f(t_n, y(t_n)) + f(t_{n+1}, y^{(j)}(t_{n+1}))], \quad j = 0, 1, \ldots \tag{13}$$

which (13) is known as iterative solution of modified Euler's method relation. To analyze the iteration and to determine conditions under which it will converge, subtract (13) from (12) to obtain

$$y(t_{n+1}) - y^{(j+1)}(t_{n+1}) = \frac{h}{2}[f(t_n, y(t_n)) + f(t_{n+1}, y^{(j)}(t_{n+1}))]. \tag{14}$$

Paper ID: MR241128145849 DOI: https://dx.doi.org/10.21275/MR241128145849

Use the Lipschitz condition in problem (8) to bound this with

$$|y(t_{n+1}) - y^{(j+1)}(t_{n+1})| \leq \frac{hK}{2} |y(t_{n+1}) - y^{(j)}(t_{n+1})| \quad j \geq 0, \tag{15}$$

thus

$$|y(t_{n+1}) - y^{(j+1)}(t_{n+1})| \leq \left(\frac{hK}{2}\right)^{j+1} |y(t_{n+1}) - y^{(0)}(t_{n+1})|. \tag{16}$$

If

$$\frac{hK}{2} \leq 1$$

then the iterates "$y^{(j)}(t_{n+1})$ will converge to $y(t_{n+1})$ as $j \to \infty$, and the computation of y_{n+1} from y_n contains a truncation error of $O(h^3)$", for more details see [1].

3 Modified Euler's Method for Numerical Solution of FDEs

Let $Y = [Y_1, Y_2]$ be the exact solution and $y = [y_1, y_2]$ be the approximated solution of the initial value equation(1) by using the one-step modified method. Let,

$$[Y(t)]_r = [Y_1(t;r), Y_2(t;r)], \; [y(t)]_r = [y_1(t;r), y_2(t;r)].$$

"Also we note that throughout each integration step, the value of r is unchanged. The exact and approximated solution at t_n are denoted by"

$$[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)], \; [y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)] \; (0 \leq n \leq N), \text{ respectively.}$$

The grid points at which the solution is calculated are

$$h = \frac{T - t_0}{N}, \quad t_i = t_0 + ih \quad 0 \leq i \leq N.$$

By using the modified Euler method we obtain:

$$\begin{aligned} Y_1(t_{n+1};r) = {} & Y_1(t_n;r) + \frac{h}{2} F[t_n, Y_1(t_n;r), Y_2(t_n;r)] \\ & + \frac{h}{2} F\big[t_{n+1}, Y_1(t_n;r) + hF[t_n, Y_1(t_n;r), Y_2(t_n;r)] \\ & , Y_2(t_n;r) + hG[t_n, Y_1(t_n;r), Y_2(t_n;r)]\big] \\ & + h^3 A_1(r) \end{aligned} \tag{17}$$

and

$$\begin{aligned} Y_2(t_{n+1};r) = {} & Y_2(t_n;r) + \frac{h}{2} G[t_n, Y_1(t_n;r), Y_2(t_n;r)] \\ & + \frac{h}{2} G\big[t_{n+1}, Y_1(t_n) + hF[t_n, Y_1(t_n;r), Y_2(t_n;r)] \\ & , Y_2(t_n;r) + hG[t_n, Y_1(t_n;r), Y_2(t_n;r)]\big] \\ & + h^3 A_2(r) \end{aligned} \tag{18}$$

where $A = [A_1, A_2]$, $[A]_r = [A_1(r), A_2(r)]$ and

$$[A]_r = [\frac{1}{4} f(\xi_2, Y_2(\xi_2)).f_y(t_{n+1}, \xi_3) - \frac{1}{12} f''(\xi_1, Y_1(\xi_1))]_r . \tag{19}$$

Also we have

$$\begin{aligned} y_1(t_{n+1}; r) &= y_1(t_n; r) + \frac{h}{2} F[t_n, y_1(t_n; r), y_2(t_n; r)] \\ &+ \frac{h}{2} F\big[t_{n+1}, y_1(t_n; r) + hF[t_n, y_1(t_n; r), y_2(t_n; r)] \\ &, y_2(t_n; r) + hG[t_n, y_1(t_n; r), y_2(t_n; r)]\big] \end{aligned} \tag{20}$$

and

$$\begin{aligned} y_2(t_{n+1}; r) &= y_2(t_n; r) + \frac{h}{2} G[t_n, y_1(t_n; r), y_2(t_n; r)] \\ &+ \frac{h}{2} G\big[t_{n+1}, y_1(t_n) + hF[t_n, y_1(t_n; r), y_2(t_n; r)] \\ &, y_2(t_n; r) + hG[t_n, y_1(t_n; r), y_2(t_n; r)]\big] \end{aligned} \tag{21}$$

We will next demonstrate that whenever h → 0, the y1(t; r) and y2(t; r) described in the earlier manner converge to Y1(t; r) and Y2(t; r), respectively. We first review the following lemmas in order to demonstrate these claims.

Lemma 3.1 *Let the sequence of numbers* $\{W_n\}_{n=0}^{N}$ *satisfy*

$$|W_{n+1}| \leq A|W_n| + B, \ 0 \leq n \leq N - 1,$$

For the specified positive constants A and B. Then

$$|W_n| \leq A^n |W_0| + B\frac{A^n - 1}{A - 1}, \ 0 \leq n \leq N.$$

Proof. See[10]. *2*

Lemma 3.2 *Let the sequence of numbers* $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ *satisfy*

$$|W_{n+1}| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \leq |V_n| + A \max\{|W_n|, |V_n|\} + B,$$

for the given positive constants A *and* B*. Then, denoting*

$$U_n = |W_n| + |V_n|, \quad 0 \leq n \leq N,$$

we have

$$U_n \leq \bar{A}^n U_0 + \bar{B}\frac{\bar{A}^n - 1}{\bar{A} - 1}, \quad 0 \leq n \leq N,$$

where $\bar{A} = 1 + 2A$ *and* $\bar{B} = 2B$.

Proof. See[10].

Our following observations confirmed the point-wise convergence of the modified "Euler approximations" to the exact answer. Let $F[t, u, v]$ and $G[t, u, v]$ be the functions which are given by the equations (6), (7) where u and v are constants and $u \le v$. Thus the domain of F and G are defined as the following:

$$K = \{(t, u, v) \mid t_0 \le t \le T, -\infty < u \le v, -\infty < v < \infty\}.$$

In what follows, we shall prove the convergence theorem using the aforementioned symbols.

Theorem 3.3 *Assume that the partial derivatives of F (t, u, v) and G (t, u, v) are constrained over RF and that they belong to C1(RF). The numerical solutions of (22) and (23) then converge uniformly in t to the precise solutions Y1(t; r) and Y2(t; r) for arbitrarily defined r, 0 < r ≤ 1.*

Proof. It is sufficient to show

$$\lim_{h\to 0} y_1(t_N; r) = Y_1(t_N; r), \qquad \lim_{h\to 0} y_2(t_N; r) = Y_2(t_N; r)$$

where "$t_N = T$. Let $W_n = Y_1(t_n; r) - y_1(t_n; r)$, $V_n = Y_2(t_n; r) - y_2(t_n; r)$, by using the equations (19), (20), (22) and (23)", we get:

$$|W_{n+1}| \le |W_n| + Lh \max\{|W_n|, |V_n|\} + Lh[2Lh \max\{|W_n|, |V_n|\} + \max\{|W_n|, |V_n|\}] + h^3 M_1,$$

$$|V_{n+1}| \le |V_n| + Lh \max\{|W_n|, |V_n|\} + Lh[2Lh \max\{|W_n|, |V_n|\} + \max\{|W_n|, |V_n|\}] + h^3 M_2,$$

where M_1, M_2 are upper bound for $A_1(r)$, $A_2(r)$ respectively. Hence,

$$|W_{n+1}| \le |W_n| + Lh\{1 + (1 + 2Lh)\} \max\{|W_n|, |V_n|\} + h^3 M,$$

$$|V_{n+1}| \le |V_n| + Lh\{1 + (1 + 2Lh)\} \max\{|W_n|, |V_n|\} + h^3 M,$$

where $M = \max\{M_1, M_2\}$, and $L > 0$ is a bound for the partial derivatives of F and G. Therefore from Lemma 5.2, we obtain

$$|W_n| \le (1 + 2Lh)^{2n} |U_0| + 2h^3 M \frac{(1 + 2Lh)^{2n} - 1}{(1 + 2Lh)^2 - 1},$$

$$|V_n| \le (1 + 2Lh)^{2n} |U_0| + 2h^3 M \frac{(1 + 2Lh)^{2n} - 1}{(1 + 2Lh)^2 - 1},$$

where $|U_0| = |W_0| + |V_0|$. In particular,

$$|W_N| \le (1 + 2Lh)^{2N} |U_0| + 2h^3 M \frac{(1 + 2Lh)^{\frac{2(T - t_0)}{h}} - 1}{(1 + 2Lh)^2 - 1},$$

Paper ID: MR241128145849 DOI: https://dx.doi.org/10.21275/MR241128145849

$$|V_N| \leq (1+2Lh)^{2N}|U_0| + 2h^3M\frac{(1+2Lh)^{\frac{2(T-t_0)}{h}}-1}{(1+2Lh)^2-1},$$

since $W_0 = V_0 = 0$, we have

$$|W_N| \leq M\frac{e^{4L(T-t_0)}-1}{2L(1+hL)}h^2, \qquad |V_N| \leq M\frac{e^{4L(T-t_0)}-1}{2L(1+hL)}h^2,$$

Thus, if $h \to 0$, we conclude $W_N \to 0$ and $V_N \to 0$, which completes the proof.2

By using modified Euler method (14), we obtain:

$$\begin{aligned} y_1(t_{n+1};r) &= y_1(t_n;r) + \frac{h}{2}\Big[F(t_n, y_1(t_n;r), y_2(t_n;r)) + F(t_{n+1}, y_1(t_{n+1};r), y_2(t_{n+1};r))\Big], \\ y_2(t_{n+1};r) &= y_2(t_n;r) + \frac{h}{2}\Big[G(t_n, y_1(t_n;r), y_2(t_n;r)) + G(t_{n+1}, y_1(t_{n+1};r), y_2(t_{n+1};r))\Big], \end{aligned} \tag{22}$$

and from (11), we have $y_1(t_{n+1};r)$ and $y_2(t_{n+1};r)$ in right side of above equations as follows:

$$\begin{aligned} y_1(t_{n+1};r) &= y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)], \\ y_2(t_{n+1};r) &= y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)]. \end{aligned} \tag{23}$$

From section 5, we consider initial guesses,

$$\begin{aligned} y_1^{(0)}(t_{n+1};r) &= y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)], \\ y_2^{(0)}(t_{n+1};r) &= y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)], \end{aligned} \tag{24}$$

for the iterative solutions below, respectively:

$$\begin{aligned} y_1^{(j+1)}(t_{n+1};r) &= y_1(t_n;r) + \frac{h}{2}\Big[F(t_n, y_1(t_n;r), y_2(t_n;r)) + F(t_{n+1}, y_1^{(j)}(t_{n+1};r), y_2^{(j)}(t_{n+1};r))\Big], \\ y_2^{(j+1)}(t_{n+1};r) &= y_2(t_n;r) + \frac{h}{2}\Big[G(t_n, y_1(t_n;r), y_2(t_n;r)) + G(t_{n+1}, y_1^{(j)}(t_{n+1};r), y_2^{(j)}(t_{n+1};r))\Big]. \end{aligned} \tag{25}$$

In order to show the next crucial theorem, the following lemma must be proven.

Lemma 3.4 *Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^1(\mathbb{R}_F)$ and the partial derivatives of F and G be bounded over $\mathbb{R}_F$. Then for arbitrarily fixed r,* $0 \leq r \leq 1$,

$$D(y(t_{n+1}), y^{(0)}(t_{n+1})) \leq h^2L(1+2C),$$

where L is a bound of partial derivatives of F and G, and
$C = \max\{|G[t_N, y_1(t_N;r), y_2(t_{N-1};r)]| \mid r \in [0,1]\} < \infty.$

Proof. By substituting (23) in (22) and subtraction obtained equation from (24), we get,

$$
\begin{aligned}
&y_1(t_{n+1};r) - y_1^{(0)}(t_{n+1};r) \\
&= \frac{h}{2}\Big\{F\Big[t_{n+1},\, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)],\, y_2(t_n;r) \\
&\quad + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] - F[t_n, y_1(t_n;r), y_2(t_n;r)]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&y_2(t_{n+1};r) - y_2^{(0)}(t_{n+1};r) \\
&= \frac{h}{2}\Big\{G\Big[t_{n+1},\, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)],\, y_2(t_n;r) \\
&\quad + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] - G[t_n, y_1(t_n;r), y_2(t_n;r)]\Big\},
\end{aligned}
$$

and from those, we can get,

$$
\begin{aligned}
&y_1(t_{n+1};r) - y_1^{(0)}(t_{n+1};r) \\
&= \frac{h}{2}\Big\{F\Big[t_{n+1},\, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)],\, y_2(t_n;r) \\
&\quad + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
&\quad - F\Big[t_n,\, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)],\, y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
&\quad + F\Big[t_n,\, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)],\, y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
&\quad - F[t_n, y_1(t_n;r), y_2(t_n;r)]\Big\},
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&y_2(t_{n+1};r) - y_2^{(0)}(t_{n+1};r) \\
&= \frac{h}{2}\Big\{G\Big[t_{n+1},\, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)],\, y_2(t_n;r) \\
&\quad + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
&\quad - G\Big[t_n,\, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)],\, y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
&\quad + G\Big[t_n,\, y_1(t_n;r) + hF[t_n, y_1(t_n;r), y_2(t_n;r)],\, y_2(t_n;r) + hG[t_n, y_1(t_n;r), y_2(t_n;r)]\Big] \\
&\quad - G[t_n, y_1(t_n;r), y_2(t_n;r)]\Big\}.
\end{aligned}
\tag{27}
$$

The following relations are derived by applying the mean value theorem to the partial derivatives of F and G, when L > 0 (26) and

$$
\begin{aligned}
&|y_1(t_{n+1};r) - y_1^{(0)}(t_{n+1};r)| \\
&\le \frac{h^2L}{2}\Big\{1 + |F[t_n, y_1(t_n;r), y_2(t_n;r)]| + |G[t_n, y_1(t_n;r), y_2(t_n;r)]|\Big\}, \\
&\le \frac{h^2L}{2}\{1 + 2|G(t_n, y_1(t_n;r), y_2(t_n;r))|\}, \\
&|y_2(t_{n+1};r) - y_2^{(0)}(t_{n+1};r)| \\
&\le \frac{h^2L}{2}\Big\{1 + |F[t_n, y_1(t_n;r), y_2(t_n;r)]| + |G[t_n, y_1(t_n;r), y_2(t_n;r)]|\Big\}, \\
&\le \frac{h^2L}{2}\{1 + 2|G(t_n, y_1(t_n;r), y_2(t_n;r))|\}.
\end{aligned} \tag{28}
$$

In particular,

$$
\begin{aligned}
|y_1(t_N);r) - y_1^{(0)}(t_N;r)| &\le \frac{h^2L}{2}(1+2C), \\
|y_2(t_N;r) - y_2^{(0)}(t_N;r)| &\le \frac{h^2L}{2}(1+2C),
\end{aligned}
$$

by adding two inequalities, one obtains,

$$|y_1(t_N);r) - y_1^{(0)}(t_N;r)| + |y_2(t_N;r) - y_2^{(0)}(t_N;r)| \le h^2L(1+2C).$$

Hence

$$D(y(t_N), y^{(0)}(t_N)) \le h^2L(1+2C), \tag{29}$$

This completes the proof. 2

Theorem 3.5 *" Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F and $2Lh < 1$. Then for arbitrarily fixed $0 \le r \le 1$, the iterative numerical solutions of $y^{(j)}(t_n;r)$ and $y_1^{(j)}(t_n;r)$ converge to the numerical solutions $y_1(t_n;r)$ and $y_2(t_n;r)$ in $t_0 \le t_n \le t_N$, when $j \to \infty$."*

Proof. It is sufficient to show

$$\lim_{j\to\infty} y_1^{(j)}(t_N;r) = y_1(t_N;r), \qquad \lim_{j\to\infty} y_2^{(j)}(t_N;r) = y_2(t_N;r)$$

where $t_N = T$. For $n = 0, 1, \ldots, N-1$, By using the equations (22) and (25), we get:

$$
\begin{aligned}
&y_1(t_{n+1};r) - y_1^{(j+1)}(t_{n+1};r) \\
&= \frac{h}{2}\{F[t_{n+1}, y_1(t_{n+1};r), y_2(t_{n+1};r)] - F[t_{n+1}, y_1^{(j)}(t_{n+1};r), y_2^{(j)}(t_{n+1};r)]\}, \\
&y_2(t_{n+1};r) - y_2^{(j+1)}(t_{n+1};r) \\
&= \frac{h}{2}\{G[t_{n+1}, y_1(t_{n+1};r), y_2(t_{n+1};r)] - G[t_{n+1}, y_1^{(j)}(t_{n+1};r), y_2^{(j)}(t_{n+1};r)]\}.
\end{aligned} \tag{30}
$$

Let L > 0 be a bound for the partial derivatives of F and G; the following relations are derived from the application of the mean value theorem to (30):

$$|y_1(t_{n+1};r) - y_1^{(j+1)}(t_{n+1};r)| \leq \frac{Lh}{2}\{|y_1(t_{n+1};r) - y_1^{(j)}(t_{n+1};r)| + |y_2(t_{n+1};r) - y_2^{(j)}(t_{n+1};r)|\},$$

$$|y_1(t_{n+1};r) - y_1^{(j+1)}(t_{n+1};r)| \leq \frac{Lh}{2}\{|y_1(t_{n+1};r) - y_1^{(j)}(t_{n+1};r)| + |y_2(t_{n+1};r) - y_2^{(j)}(t_{n+1};r)|\}. \quad (31)$$

Consequently, as derived from Definition D, the Hausdorff distance, as discussed in section 2, will yield:

$$|y_1(t_{n+1};r) - y_1^{(j+1)}(t_{n+1};r)| \leq LhD(y(t_{n+1}), y^{(j)}(t_{n+1})),$$

$$|y_2(t_{n+1};r) - y_2^{(j+1)}(t_{n+1};r)| \leq LhD(y(t_{n+1}), y^{(j)}(t_{n+1})).$$

Hence, “adding two inequalities gives”,

$$D(y(t_{n+1}), y^{(j+1)}(t_{n+1})) \leq 2LhD(y(t_{n+1}), y^{(j)}(t_{n+1}))$$

$$\vdots$$

$$D(y(t_{n+1}), y^{(j+1)}(t_{n+1})) \leq (2Lh)^{j+1}D(y(t_{n+1}), y^{(0)}(t_{n+1})).$$

Using lemma 3.4 in special case, we get:

$$D(y(t_N), y^{(j+1)}(t_N)) \leq \frac{1}{2}(2Lh)^{j+2}h(1+2C).$$

The desired result finally follows from condition $2Lh \leq 1$,

$$\lim_{j\to\infty} D([y(t_N)]_r, [y^{(j)}(t_N)]_r) = 0.$$

The proof is complete. 2

4 Numerical Results

There are two numerical examples that we will provide in this section. We have created an error table for each of the cases in order to determine the degree of precision that exists between our numerical solution and the theoretical precise solution. The errors are derived from the equation “$D[Y(t;r), y(t;r)]$ for $t = t_N$, $r \in [0, 1]$.”

In addition to the “convergence theorem”, the numerical results demonstrate that errors are reduced as h decreases. The precise solutions and the approximated solutions for example 1 and example 2 are illustrated in figure 1 and figure 2, respectively, employing the Euler method alongside the proposed method “(Mod.Euler method)”.

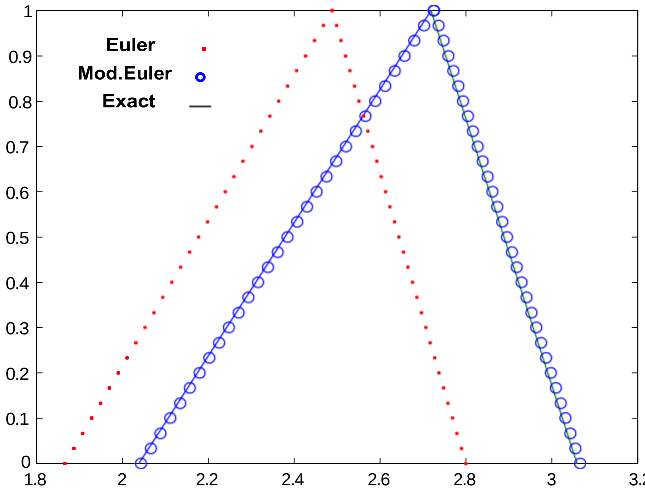

Figure 1: h=0.2

Example 4.1 *Consider the initial value problem* [10]

$$y'(t) = y(t), \qquad t \in [0,1]$$
$$y(0) = (0.75 + 0.25r, 1.125 - 0.125r)$$

The exact solution at $t = 1$ *is given by*

$$Y(1;r) = [(0.75 + 0.25r)e, \ (1.125 - 0.125r)e], \qquad 0 \le r \le 1.$$

Using iterative solution of modified Euler's method, we have

$$y_1(0;r) = 0.25 + 0.25r, \ \ y_2(0;r) = 1.125 - 0.125r,$$

and by

$${}_1y^{(0)}(t_{i+1};r) = y_1(t_i;r) + hy_1(t_i;r),$$
$${}_2y^{(0)}(t_{i+1};r) = y_2(t_i;r) + hy_2(t_i;r),$$

where $i = 0, 1, \ldots, N-1$ and $h = \frac{1}{N}$. Now, using these equations as an initial guess for following iterative solutions, respectively,

$${}_1y^{(j)}(t_{i+1};r) = y_1(t_i;r) + \frac{h}{2}[y_1(t_i;r) + {}_1y^{(j-1)}(t_{i+1};r)],$$
$${}_2y^{(j)}(t_{i+1};r) = y_2(t_i;r) + \frac{h}{2}[y_2(t_i;r) + {}_2y^{(j-1)}(t_{i+1};r)],$$

where $j = 1, 2, 3$. Thus we have $y_1(t_i;r) = {}_1y^{(3)}(t_i;r)$ and $y_2(t_i;r) = {}_2y^{(3)}(t_i;r)$, for $i = 1, \ldots, N$.
Therefore, $Y_1(1;r) \approx {}_1y^{(3)}(1;r)$ and $Y_2(1;r) \approx {}_2y^{(3)}(1;r)$ are obtained. Table 1 shows estimation of error for different values of $r \in [0,1]$ and h.

Table 1

h / r	0.1	0.01	0.001	.0001
0	0.0025350660	0.0000254824	0.0000002548	0.0000000025
0.2	0.0024787331	0.0000249162	0.0000002491	0.0000000024
0.4	0.0024223964	0.0000243499	0.0000002435	0.0000000024
0.6	0.0023660616	0.0000237836	0.0000002378	0.0000000023
0.8	0.0023097268	0.0000232173	0.0000002321	0.0000000023
1	0.0022533920	0.0000226510	0.0000002265	0.0000000022

Example 4.2 *Consider the fuzzy initial value problem*

$$y'(t) = k_1 y^2(t) + k_2, \qquad y(0) = 0,$$

where $k_j > 0 (j = 1, 2)$ *are triangular fuzzy numbers.*

The exact solution is given by

$$Y_1(t; r) = l_1(r) \tan(w_1(r)t),$$
$$Y_2(t; r) = l_2(r) \tan(w_2(r)t),$$

with

$$l_1(r) = \sqrt{k_{2,1}(r)/k_{1,1}(r)}, \qquad l_2(r) = \sqrt{k_{2,2}(r)/k_{1,2}(r)},$$
$$w_1(r) = \sqrt{k_{1,1}(r)k_{2,1}(r)}, \qquad w_2(r) = \sqrt{k_{1,2}(r)k_{2,2}(r)},$$

where

$$[k_1]_r = [k_{1,1}(r), k_{1,2}(r)] \text{ and } [k_2]_r = [k_{2,1}(r), k_{2,2}(r)],$$
$$k_{1,1}(r) = 0.5 + 0.5r, \qquad k_{1,2}(r) = 1.5 - 0.5r,$$
$$k_{2,1}(r) = 0.75 + 0.25r, \qquad k_{2,2}(r) = 1.25 - 0.25r.$$

Now by using equations below

$$y_1(0; r) = y_2(0; r) = 0,$$
$$y_1^{(0)}(t_{i+1}; r) = y_1(t_i; r) + h(k_{11} y_1^2(t_i; r) + k_{21}),$$
$$y_2^{(0)}(t_{i+1}; r) = y_2(t_i; r) + h(k_{12} y_2^2(t_i; r) + k_{22}),$$

for $i = 0, 1, \ldots, N - 1$ and $h = \frac{1}{N}$, as an initial guess for following iterative solutions, respectively,

$$y_1^{(j)}(t_{i+1}; r) = y_1(t_i; r) + \frac{h}{2}[k_{11} y_1^2(t_i; r) + k_{11}(y_1^{(j-1)}(t_{i+1}; r))^2 + 2k_{21}]$$
$$y_2^{(j)}(t_{i+1}; r) = y_2(t_i; r) + \frac{h}{2}[k_{12} y_2^2(t_i; r) + k_{12}(y_2^{(j-1)}(t_{i+1}; r))^2 + 2k_{22}]$$

where $j = 1, 2, 3$. Similar to example 6.1,we have $y_1(t_i; r) = y_1^{(3)}(t_i; r)$ and $y_2(t_i; r) = y_2^{(3)}(t_i; r)$, for $i = 1, \ldots, N$.

Paper ID: MR241128145849 DOI: https://dx.doi.org/10.21275/MR241128145849

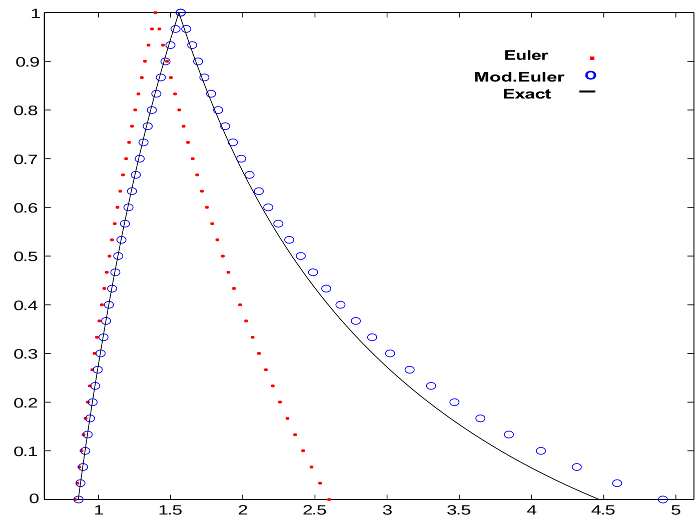

Figure 2: h=0.1

Therefore, $Y_1(1;r) \approx y_1^{(3)}(1;r)$ and $Y_2(1;r) \approx y_2^{(3)}(1;r)$. Table 2 shows estimation of error for different values of $r \in [0,1]$ and h.

Table 2

h / r	0.1	0.01	0.001	.0001
0	0.4417099428	0.0079388827	0.0000845005	0.0000008504
0.2	0.18106314062	0.0027073124	0.0000282513	0.0000001172
0.4	0.0847937757	0.0011335243	0.0000116861	0.0000001172
0.6	0.0433920492	0.0005375096	0.0000054962	0.0000000550
0.8	0.0235983073	0.0002766096	0.0000028118	0.0000000281
1	0.0133874352	0.0001505383	0.0000015235	0.0000000152

5 Conclusion

This study presents the application of an iterative approach utilizing modified Euler's method for the numerical resolution of "fuzzy differential equations". It is clear that the approach presented in this paper, characterized by an O(h3) complexity, surpasses "Euler's method", which is defined by an O(h) complexity [10].

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Paper ID: MR241128145849 DOI: https://dx.doi.org/10.21275/MR241128145849