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Few Identities of Rogers-Ramanujan Type and Some q-Product Identities Arising from Theta Function Identities

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Abstract: ***In this paper, some identities of the Rogers-Ramanujan Type modulo 5, 7, 10, 12 and 14 has been derived analytically from the generalised Watson's q −analogue of Whipple's theorem. We also have derived some q −product Identities from the generalised Schroter's formula by using Ramanujan's Theta function Identities.***

Keywords: Watson's q −analogue of Whipple's theorem, Ramanujan's theta function, generalised Schroter's formula, Jacobis Triple Product Identity.

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1. Introduction

Definitions, preliminary results and notations:

Throughout this paper, we assume $|q|$<1and use the standard notations

$(a;q)_0 = 1$

$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$, *for n≥1*

and,

$(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^k)$.

It follows that $\quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorial is defined by

$$(a_1,a_2,\ldots..a_m;q)_n = (a_1;q)_n(a_2;q)_n \ldots.(a_m;q)_n$$
$$(a_1,a_2,\ldots..a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty \ldots.(a_m;q)_\infty$$

For convenience of our simplification, we use some simple results such as:

$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$

$(a;q)_n\,(-a;q)_n = (a^2;q^2)_n$, $0\leq n<\infty$

$(q;q^2)_n\,(q^2;q^2)_n = (q;q)_{2n}$

$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$, $-\infty<n<\infty$

For $|ab|<1$, Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2} \qquad (2.1)$$

The Jacobi Triple Product Identity (See [7], p.21, Theorem 2.8) is given by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \qquad (2.2)$$

Another form of the identity (2.2) is the following:

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)\frac{n(n+1)}{2}-in} = \sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)\frac{n(n+1)}{2}-in}(1-q^{(2n+1)i}) = \prod_{n=0}^{\infty}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i}) \qquad (2.3)$$

provided $|ab|<1$.

Some special cases of $f(a,b)$ as listed in ([5], Equations (1.4.9)- (1.4.11), p. 17) are as follows

For $|q|<1$,

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}λλ = (-q;q^2)_\infty^2(q^2;q^2)_\infty \qquad (2.4)$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \qquad (2.5)$$

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \qquad (2.6)$$

We also mention the two more results related to $f(a,b)$: (See [1], Entry 30, p. 46])

$$f(a,b)+f(-a,-b) = 2\,f(a^3b,ab^3) \qquad (2.7)$$

and

$$f(a,b)-f(-a,-b) = 2a\,f(b/a,a^5b^3) \qquad (2.8)$$

The functions φ and ψ as mentioned in (2.4) and (2.5) satisfies the following relations (See [1], Entry 25, p. 39])

$$\varphi(q)+\varphi(-q) = 2\varphi(q^4) \qquad (2.9)$$
$$\varphi(q)-\varphi(-q) = 4q\,\psi(q^8) \qquad (2.10)$$
$$\varphi(q)\varphi(-q) = \varphi^2(-q^2) \qquad (2.11)$$

In Chapter 11 of [8], many Identities of Rogers-Ramanujan Type has been derived by G. E. Andrews and B. C. Berndt and majority of them can be proved as special limiting cases of the Watson's q −analogue of Whipple's theorem ([9], p. 242, eqn. (III.18)).

If $\alpha,\beta,\gamma,\delta$ and ε are any complex numbers such that $\beta\gamma\delta\varepsilon \neq 0$ and if N is any non-negative integer, then

$$ {}_8\phi_7\left(\begin{matrix}\alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \quad \beta, \quad \gamma, \quad \delta, \quad \varepsilon, \quad q^{-N}; q, \frac{\alpha^2 q^{N+2}}{\beta\gamma\delta\varepsilon} \\ \sqrt{\alpha}, \ -\sqrt{\alpha}, \ \frac{\alpha q}{\beta}, \frac{\alpha q}{\gamma}, \frac{\alpha q}{\delta}, \frac{\alpha q}{\varepsilon}, \alpha q^{N+1}\end{matrix}\right) = \frac{(aq)_N \ (\frac{aq}{\delta\epsilon})_N}{(\frac{aq}{\delta})_N(\frac{aq}{\epsilon})_N} \ {}_4\phi_3\left(\begin{matrix}\frac{\alpha q}{\beta\gamma}, \delta, \varepsilon, \quad q^{-N}; q, q \\ \frac{\alpha q}{\beta}, \frac{\alpha q}{\gamma}, \frac{\delta\varepsilon q^{-N}}{\alpha}\end{matrix}\right) \quad (2.12) $$

Two more results of this nature were proved by W. N. Bailey ([7], eqn. (6.1) and (6.3)),

$$ \sum_{n=0}^{\infty}\frac{(\rho_1;q)_n(\rho_2;q)_n(\frac{aq}{f};q^2)_n}{(q;q)_n(aq;q^2)_n(aq/f;q)_n}.\left(\frac{aq}{\rho_1\rho_2}\right)^n = \frac{(\frac{aq}{\rho_1};q)_\infty(\frac{aq}{\rho_2};q)_\infty}{(aq;q)_\infty(aq/\rho_1\rho_2;q)_\infty} \times\left(1+\sum_{n=1}^{\infty}\frac{(aq^2;q^2)_{n-1}(f;q^2)_n(\rho_1;q)_{2n}(\rho_2;q)_{2n}(1-aq^{4n})}{(q^2;q^2)_n(\frac{aq^2}{f};q^2)_n(\frac{aq}{\rho_1};q)_{2n}(\frac{aq}{\rho_2};q)_{2n}}\left(\frac{a^3}{\rho_1^2\rho_2^2 f}\right)^n q^{2n^2+2n}\right) \quad (2.13) $$

and

$$ \sum_{n=0}^{\infty}\frac{(\rho_1;q^2)_n(\rho_2;q^2)_n(-\frac{aq}{b};q)_{2n}}{(q^2;q^2)_n(a^2q^2/b^2;q^2)_n(-aq;q)_{2n}}.\left(\frac{a^2q^2}{\rho_1\rho_2}\right)^n = \frac{(\frac{a^2q^2}{\rho_1};q^2)_\infty(\frac{a^2q^2}{\rho_2};q^2)_\infty}{(a^2q^2;q^2)_\infty(a^2q^2/\rho_1\rho_2;q^2)_\infty} \times\left(1+\sum_{n=1}^{\infty}\frac{(aq;q)_{n-1}(b;q)_n(\rho_1;q^2)_n(\rho_2;q^2)_n(1-aq^{2n})}{(q;q)_n(\frac{aq}{b};q)_n(\frac{a^2q^2}{\rho_1};q^2)_n(\frac{a^2q^2}{\rho_2};q^2)_n}\left(\frac{a^3q^2}{\rho_1\rho_2 b}\right)^n q^{n^2}\right) \quad (2.14) $$

Another interesting identity known as Schröter identity along with one of its extensions has been mentioned here. One special case of this extension will be needed in the subsequent section.

Here, the notation

$$[a;q^{2j}]_\infty$$

is used to denote the triple product

$$(a,\frac{q^{2j}}{a},q^{2j};q^{2j})_\infty$$

***Schröter's* Identity:** (See [2], p. 111])

$$ [-q^\mu a;q^{2\mu}]_\infty[-q^\gamma b;q^{2\gamma}]_\infty = \sum_{j=0}^{\mu+\gamma-1} q^{\mu j^2}a^j \left[\frac{-q^{\mu+\gamma+2\mu j}.a}{b};q^{2(\mu+\gamma)}\right]_\infty \times[-q^{(\mu+\gamma+2j)\mu\gamma}a^\gamma b^\mu;q^{2(\mu+\gamma)\mu\gamma}]_\infty \quad (2.15) $$

Lemma: (An extension of Schröter Identity, (See[10], Lemma 3.3, p-893)) Let a,b and z be non-zero complex numbers, let q be a complex number with $|q|<1$, and let $m\geq 1$ be an integer. Then

$$ [-q^\mu az;q^{2\mu}]_\infty[-q^\gamma bz;q^{2\gamma}]_\infty = \sum_{j=0}^{m(\mu+\gamma)-1} q^{\mu j^2}(az)^j \left[\frac{-q^{\mu+\gamma+2\mu j}.a}{b};q^{2(\mu+\gamma)}\right]_\infty \times [-q^{(m(\mu+\gamma+2j)\mu\gamma m}a^{m\gamma}b^{m\mu}z^{m(\mu+\gamma)};q^{2(\mu+\gamma)\mu\gamma m^2}]_\infty \quad (2.16) $$

We conclude this section by recalling the following identity that can considered as special case of Schröter Identity (See [1],p. 68)

If μ and γ are integers such that $\mu>\gamma\geq 0, then$

$$ \frac{1}{2}\{\varphi(q^{\mu+\gamma})\varphi(q^{\mu-\gamma})+\varphi(-q^{\mu+\gamma})\varphi(-q^{\mu-\gamma})\} = \sum_{m=0}^{\mu-1} q^{2\mu m^2} f\left(q^{(2\mu+4m)(\mu^2-\gamma^2)},q^{(2\mu-4m)(\mu^2-\gamma^2)}\right) f(q^{2\mu+4\gamma m},q^{2\mu-4\gamma m}) \quad (2.17) $$

2. Main Results

First we derive some transformations from the results (2.12)-(2.14) as follows:

In (2.13), replace a by aq^2, let f,ρ_1 and ρ_2 tends to ∞, we find the following transformation-

$$ \sum_{n=0}^{\infty}\frac{a^n q^{n^2+2n}}{(q;q)_n(aq^3;q^2)_n} = \frac{1}{(aq^3;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(aq^4;q^2)_{n-1}(1-aq^{4n+2})a^{3n}q^{7n^2+5n}}{(q^2;q^2)_n}\right) \quad (3.1) $$

Replace a by aq^2, set $f=-aq/b$ and let ρ_1 and ρ_2 tends to ∞ in (2.13), we obtain the following:

$$ \sum_{n=0}^{\infty}\frac{(-bq^2;q^2)_n a^n q^{n^2+2n}}{(q;q)_n(aq^3;q^2)_n(-bq^2;q)_n} = \frac{1}{(aq^3;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(aq^4;q^2)_{n-1}(-\frac{aq}{b};q^2)_n(1-aq^{4n+2})a^{2n}b^n q^{6n^2+5n}}{(q^2;q^2)_n(-bq^3;q^2)_n}\right) \quad (3.2) $$

In the same way (2.13) yields another transformation (3.3) upon replace of a by aq, ρ_2 by $-\frac{aq}{b}$ and for $f\to\infty$, $\rho_1\to\infty$.

$$ \sum_{n=0}^{\infty}\frac{(-\frac{aq}{b};q)_n b^n q^{n(n+1)/2}}{(q;q)_n(aq^2;q^2)_n} = \frac{(-bq;q)_\infty}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(aq;q^2)_n(-aq/b;q)_{2n}(1-aq^{4n+1})a^n b^{2n}q^{5n^2+n}}{(q^2;q^2)_n(-bq;q)_{2n}} \quad (3.3) $$

Identities of Rogers-Ramanujan type:

Identities Related to Modulo 14:

The transformation (3.1) for $a=q^{-2}$ yeilds

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n(q;q^2)_n}=\frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(q^2;q^2)_{n-1}(1-q^{4n})\,q^{7n^2-n}}{(q^2;q^2)_n}\right)$$
$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\,q^{7n^2-n}$$
$$=\frac{(q^{14};q^{14})_\infty\,(q^6;q^{14})_\infty(q^8;q^{14})_\infty}{(q;q)_\infty} \quad (3.4)$$

The transformation (3.1) for $a=1$, q^2 and q^{-2} successively yields

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q;q)_n(q;q^2)_{n+1}}=\sum_{n=-\infty}^{\infty}(-1)^n\,q^{7n^2+5n}=\frac{(q^{14};q^{14})_\infty\,(q^2;q^{14})_\infty(q^{12};q^{14})_\infty}{(q;q)_\infty} \quad (3.5)$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+4n}}{(q;q)_n(q;q^2)_{n+2}}=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n(1-q^{2n+2})(1-q^{4n+4})\,q^{7n^2+11n}$$
$$=\frac{(q^{14};q^{14})_\infty\,(q^4;q^{14})_\infty(q^{10};q^{14})_\infty}{(q;q)_\infty}+q^{-4}.\ \frac{(q^{14};q^{14})_\infty\,(q^6;q^{14})_\infty(q^8;q^{14})_\infty}{(q;q)_\infty} \quad (3.6)$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n(q;q^2)_n}=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\,q^{7n^2-n}$$
$$=\frac{(q^6;q^{14})_\infty\,(q^8;q^{14})_\infty(q^8;q^{14})_\infty}{(q;q)_\infty} \quad (3.7)$$

The identity (3.7) appears in Slatter's list [11] as (61)

Identities Related to Modulo 12:

Setting $a=1, b=\frac{1}{q}$ in (3.2), we get

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n\,q^{n^2+2n}}{(q;q)_{2n+1}}=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{6n^2-4n}$$
$$=\frac{(q^2;q^{12})_\infty\,(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty} \quad (3.8)$$

The identity (3.8) appears in Slatter's list [11] as (50)

Setting $a=q^{-2}, b=q^{-2}$ in (3.2), we get

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_n\,q^{n^2}}{(q;q)_n(q;q^2)_n(-1;q)_n}=\frac{(q^5;q^{12})_\infty\,(q^7;q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty} \quad (3.9)$$

Identities Related to Modulo 10:

The transformation (3.3) for $a=q, b=q$ yeilds,

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n\,q^{n(n+3)/2}}{(q;q)_n(q;q^2)_{n+1}}=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(-q;q)_{2n}\,(1-q^{4n+2})\,q^{5n^2+4n}}{(-q^2;q)_{2n}}$$

which upon using ([12], eq. (43), p.156), gives

$$\sum_{n=0}^{\infty}(-1)^n\,(1-q^{2n+1})\,q^{5n^2+4n}=\frac{(q^{10};q^{10})_\infty\,(q^1;q^{10})_\infty(q^9;q^{10})_\infty(q;q)_\infty}{(-q;q)_\infty} \quad (3.10)$$

The transformation (3.3) for $a=q, b=1$ gives,

$$\sum_{n=0}^{\infty}(-1)^n(1+q^{2n+1})\,(1-q^{4n+2})\,q^{5n^2+2n}=\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q)_n\,(1+q^{n+1})\,q^{n(n+1)/2}}{(q;q)_n(q;q^2)_{n+1}}$$

Now using two equations of Slatter List ([11], eqn. (43)-(44), p. 156) in the above equation, we obtain the following identities-

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n(1+q^{2n+1})\,(1-q^{4n+2})\,q^{5n^2+2n}$$
$$=(q^3,q^7,q^{10};q^{10})_\infty+q.(q,q^9,q^{10};q^{10})_\infty \quad (3.11)$$

In (2.14), taking b,ρ_1 and ρ_2 tends to ∞, we find the following transformation

$$\sum_{n=0}^{\infty}\frac{(-\frac{aq}{b};q)_{2n}\,a^{2n}\,q^{2n^2}}{(q^2;q^2)_n(-aq;q)_{2n}}.=\frac{1}{(a^2q^2;q^2)_\infty}\times\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(aq;q)_{n-1}(b;q)_n(1-aq^{2n})a^{3n}\,q^{(7n^2-n)/2}}{(q;q)_n(\frac{aq}{b};q)_n}\right) \quad (3.12)$$

Setting $a=aq$ in (3.12) and then placing $a=1, b=-q$, we find

$$\sum_{n=0}^{\infty}\frac{(q;q)_{2n}\,q^{2n^2+2n}}{(q^2;q^2)_n(-q^2;q)_{2n}}=\frac{1}{(q^4;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\,q^{(7n^2+5n)/2}$$
$$=\frac{(q;q^7)_\infty\,(q^6;q^7)_\infty(q^7;q^7)_\infty}{(q^4;q^2)_\infty} \quad (3.13)$$

In (2.14), setting $\rho_1=-aq$ and then taking b, ρ_2 tends to ∞, we find the following identity

$$\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n\,a^n q^{n^2}}{(q^2;q^2)_n(-aq;q)_{2n}}.=\frac{(-aq;q^2)_\infty}{(a^2q^2;q^2)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(aq;q)_{n-1}(1-aq^{2n})a^{2n}q^{(5n^2-n)/2}}{(q;q)_n}\right)$$

Which for $a=1$ yeilds

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n\,q^{n^2}}{(q^2;q^2)_n(-q;q)_{2n}}.=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(q;q)_{n-1}(1-q^{2n})q^{(5n^2-n)/2}}{(q;q)_n}\right)$$
$$=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n(1+q^n)q^{(5n^2-n)/2}\right)$$
$$=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{(5n^2-n)/2}\right)$$
$$=\frac{(q^2;q^5)_\infty\,(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q^2)_\infty(q^4;q^4)_\infty} \quad (3.14)$$

3. Some q-Series Identities

In this section we have derived some q −series identities as follows-

Identity I:

$(-q;q^2)_\infty^2(q;q^2)_\infty^2(q^2;q^2)_\infty^2=(q^2;q^4)_\infty^4\,(q^4;q^4)_\infty^2$ (4.1)

Identity II:

$\varphi^2(q^4)-\varphi(q^6)\varphi(q^{12})=2q\,f(-q^2,-q^{10})_\infty.\ f(q^8,q^{20})_\infty.$ (4.2)

Identity III:

$$\frac{f(q,q^3)(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty}=f(-q^3,-q^3)_\infty\,f(q^3,q^{12})_\infty-f(-q,-q^6)_\infty[q\,f(q,q^{11})_\infty+f(q^5,q^7)_\infty] \quad (4.3)$$

Identity IV:

$$\frac{(q^8;q^8)_\infty(q^4;q^4)_\infty}{(q^{12};q^{12})_\infty(q^{96};q^{96})_\infty}\left\{\frac{(-q^2;q^2)_\infty^2}{(-q^8;q^8)_\infty^2}+\frac{(q^2;q^2)_\infty^2}{(q^8;q^8)_\infty^2}\right\}$$

$=2(-q^{48};q^{96})_\infty^2(-q^6;q^{12})_\infty^2 \quad +4q^6(\frac{(-Q;Q^2)_\infty}{(-Q^3;Q^6)_\infty}\frac{(-q^2;q^4)_\infty}{(-q^6;q^{12})_\infty}$,

where $Q=q^{16}$ (4.4)

Identity V:

$(-q^5;q^{10})_\infty^2(q^{10};q^{10})_\infty(-q^3;q^6)_\infty^2(q^6;q^{10})_\infty$

$=2(-q^{120};q^{240})_\infty^2(q^{240};q^{240})_\infty(-q^8;q^{16})_\infty^2(q^{16};q^{16})_\infty$

$+8q^{32}(q^{480};q^{480})_\infty(-q^{240};q^{240})_\infty(q^{32};q^{32})_\infty(-q^{16};q^{16})_\infty$

$+4q^8(q^{120};q^{120})_\infty(-q^{60};q^{60})_\infty(q^8;q^8)_\infty(-q^4;q^4)_\infty$ (4.5)

Proof of (4.1): Setting $m=1$, $a=1$, $b=-1$, $z=1$, $\mu=1$, $\gamma=1$ in (2.16), we find the desired identity upon some mathematical manipulation.

Proof of (4.2): Setting $m=1$, $a=1$, $b=-1$, $z=1$, $\mu=2$, $\gamma=1$, in (2.16), we find upon using (2.4) and (2.6), that

$\varphi(q^2)\,\varphi(-q^2)= \quad \varphi(-q^3)\,\varphi(q^6)-\,2q\,f(-q,-q^5)_\infty\times f(-q^2,-q^{10})_\infty$ (4.6)

Replacing q $by-q$ in (4.6), it gives

$\varphi(q^2)\,\varphi(-q^2)= \quad \varphi(q^3)\,\varphi(q^6)+2q\,f(q,q^5)_\infty\times f(-q^2,-q^{10})_\infty$ (4.7)

On adding (4.6) & (4.7), we find

$2\varphi(q^2)\,\varphi(-q^2)$

$= \quad 2\varphi(q^6)\varphi(q^{12})\,+2q\,f(-q^2,-q^{10})_\infty.[\quad f(q,q^5)_\infty-f(-q,-q^5)_\infty\,]$ (4.8)

We recall the identity ([3a], eqn. (8.5.7.7) p. 254),

$f(q,q^5)_\infty=f(q^8,q^{16})_\infty+qf(q^4,q^{20})_\infty$ (4.9)

Now, employing (4.9) in (4.8), we find the desired result.

Proof of (4.3):

Setting $a=1, b=-1, m=1, z=q$, $\mu=2$, $\gamma=1$ in (2.16), we find upon using the fact $(-q;q)_\infty$ = $1/(q;q^2)_\infty$ and mathematical manipulation, the following,

$\frac{(-q,-q^3;q^4)_\infty(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty}$

$=(q^3;q^6)_\infty(q^3;q^6)_\infty(q^6;q^6)_\infty\times(-q^9;q^{12})_\infty\;\;(-q^3;q^{12})_\infty(q^{12};q^{12})_\infty$

$-q\,(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty\times(-q;q^{12})_\infty\,(-q^{11};q^{12})_\infty(q^{12};q^{12})_\infty$

$-\,(q^5;q^6)_\infty(q^1;q^6)_\infty(q^6;q^6)_\infty\times(-q^5;q^{12})_\infty\,(-q^7;q^{12})_\infty(q^{12};q^{12})_\infty$ (4.10)

Now by using the identity (2.2), we find the result (4.3).

Proof of (4.4):

Setting $\mu=3,\gamma=1$ in (2.17), we find after mathematical manipulation,

$\frac{1}{2}\{\varphi(q^4)\varphi(q^2)+\varphi(-q^4)\varphi(-q^2)\}$

$=f(q^{48},q^{48})\,f(q^6,q^6)$+$2q^6f(q^{80},q^{16})f(q^{10},q^2)$ (4.11)

Writing (4.11) in $q-$products with the aid of (2.2) and (2.4), and them simplifying mathematically, we arrive at

$\frac{(q^8;q^8)_\infty(q^4;q^4)_\infty}{2}\{\frac{(-q^2;q^2)_\infty^2}{(-q^8;q^8)_\infty^2}+\frac{(q^2;q^2)_\infty^2}{(q^8;q^8)_\infty^2}\}$

$=(q^{12};q^{12})_\infty(q^{96};q^{96})_\infty\{(-q^{48};q^{96})_\infty^2(-q^6;q^{12})_\infty^2$

$+2q^6(\frac{(-q^{16};q^{32})_\infty}{(-q^{48};q^{96})_\infty}\frac{(-q^2;q^4)_\infty}{(-q^6;q^{12})_\infty}\}$ (4.12)

Finally, putting $Q=q^{16}$ in (4.12), we obtain the desired identity.

Proof of (4.5):

Setting $\mu=4,\gamma=1$ in (2.17), we find the following after some mathematical manipulation,

$\frac{1}{2}\{\varphi(q^5)\varphi(q^3)+\varphi(-q^5)\varphi(-q^3)\}$

$=f(q^{120},q^{120})f(q^8,q^8)$+$4q^{32}f(q^{240},q^{720})f(q^{16},q^{48})$

$+2q^8f(q^{60},q^{180})f(q^4,q^{12})$ (4.13)

Upon using (2.4) and (2.5), we obtain,

$\varphi(q^5)\varphi(q^3)+\varphi(-q^5)\varphi(-q^3)$

$=2\,\varphi(q^{120})\,\varphi(q^8)$

$+8q^{32}\,\psi(q^{240})\,\psi(q^{16})$+$4q^8\,\psi(q^{60})\psi(q^4)$ (4.14)

Writing (4.14) in $q-$products with the aid of (2.2), we arrive at the desired result.

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