An HIV / AIDS Epidemic Model with Non-Linear Saturated Incidence Rate and Treatment

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Abstract: ***In this paper, take an HIV/AIDS epidemic model with a non-linear saturated incidence rate, awareness, and treatment. The model approves that some infected individuals move from the infected class to the treatment class. First, formulate the model and find its basic reproduction number. Mathematical analyses establish that this kind of truly infectious disease is spreading globally it completely depends on the basic reproduction number R_0. When $R_0 < 1$, the disease-free equilibrium is stable locally. Whether $R_0 > 1$, the endemic equilibrium point is stable locally. Also, the study mentions that the desirable sensitive parameters that can control disease transmission are awareness and treatment of the infected individuals at an early stage. Finally, we find the model's numerical solution which support the analytical outcomes.***

Keywords: Mathematical model, Non-linear incidence, Basic reproductive number, Local Stability, Treatment, Awareness

1. Introduction

Human Immunodeficiency virus or acquired immunodeficiency syndrome is one of the most enormous infectious diseases humankind has already faced, with immense social, economic, and public health consequences. People living with HIV are near about around 8 million from 1990 to 34 million at the end of 2011(USAID, 2012 [1]. First Acquired Immunodeficiency Syndrome (AIDS) cases were recorded and announced by the Centers for Disease Control (CDC) of America in 1981 [2]. In the very next year, AIDS was defined as a disease at least moderately predictive of a defect in cell-mediated immunity occurring in a person with no known cause for diminished resistance to that disease. Such types of diseases include Kaposi's sarcoma, Pneumocystis carinii pneumonia, and serious opportunistic infections [3]. In 1986, the AIDS case was given a separate name Human Immunodeficiency Virus or HIV by the International Committee on Taxonomy of Viruses [4]. In the same year that is in 1986, the first known HIV cases were reported in India among female sex workers in Chennai. In the next year around 135 new cases were reported [5]. Mathematical models play a serious role in the study of the transmission and treatment dynamics of HIV/AIDS, some models are introduced in [6-10]. Many researchers' various ideas have been added to modeling frameworks from the initial models of May and Anderson [11-15]. Researchers use mathematical models to contribute some ideas to decrease HIV/AIDS disease [16]. Due to the difficulty in treating HIV, infection, preventing infection is a key aim in restraining the AIDS pandemic, with health organizations promoting attempts to slow the spread of the virus. Usually, HIV is spread through unprotected sexual contact and from mother to child at birth and through breast milk [17]. Lack of awareness and treating the infected individuals as early as possible can spread the disease, rapidly [18-19]. In terms of HIV treatment, established from this study that the use of antiretroviral (ARVs) and increasing awareness to control the disease transmission. This paper tries to use a nonlinear saturated incidence rate to study the dynamics of HIV disease with treatment awareness. The model is solved analytically and using a numerical recipe to justify it.

2. Mathematical Model

In this study consider the total human population is constant. Susceptible population class increases at a rate R_s. The total population is divided into four sub-classes of the relevant population i.e. the susceptible population S (t), infected people I(t), fully bloom AIDS population A(t), and Treated class T(t). Let β be the contract rate between Susceptible and HIV infected, α be the disease-induced death rate due to treatment, α_1 and α_2 are two positive constants, d mortality rate of the adults class, λ_i the rate at which the infected class is recruited into the treated class, θ_i the rate at which the infected class is recruited into the AIDS class, σ_a be the treatment rate of AIDS class and δ_a be the awareness rate i.e. something which is related to disease transmission. With these consideration the model is

$$\begin{aligned}
\frac{dS}{dt} &= R_s - \frac{\beta SI}{1+\alpha_1 I+\alpha_2 I^2} - (d+\delta_a)S,\\
\frac{dI}{dt} &= \frac{\beta SI}{1+\alpha_1 I+\alpha_2 I^2} - (\lambda_i+\theta_i+d)I,\\
\frac{dA}{dt} &= \theta_i I - (d+\sigma_a)A, \qquad (1)\\
\frac{dT}{dt} &= \lambda_i I + \sigma_a A - (d+\alpha)T.
\end{aligned}$$

3. Invariant Region

Theorem 3.0.1 All possible solutions of the system (1) are positively contained in Γ_{ir}.

Proof: Choose Γ_{ir} =(S, I,A,T)∈ R^4 be the set of all positive solutions of the model (1). From the system (1),

$$\frac{dN}{dt} \le R_s - dN(t).$$

Hence, $\lim_{t\to\infty} sup(S+I+A+T) \le \frac{R_s}{d}$, so the required invariant regions is

$\Gamma_{ir} = \{(S,\ I,\ A,\ T)$: S+I+A+T$\le\frac{R_s}{d}, S \ge 0, I \ge 0, A \ge 0, T \ge 0\}$.

Hence wherever starting the solution set in the interior of the vector field Γ_{ir} it also remains in the boundary Γ_{ir} when S+I+A+T$\leq\frac{R_s}{d}$ and is positively invariant.

3.1 Positivity of Solutions

Due to consideration of the human population, it is necessary to establish all the solutions of the system (1) is positive for all time.

Theorem 3.1.1Suppose $\Gamma_{ir} = \{(S,\ I,\ A,\ T)$: S+I+A+T$\leq\frac{R_s}{d}$, $S \geq 0, I \geq 0, A \geq 0, T \geq 0\}$, then all the solutions S(t), I(t), A(t), T(t} of the system (1) are positive for all t$\geq$ 0.

Proof: Taking the first equation,

$$\frac{dS}{dt} = R_s - \frac{\beta SI}{1 + \alpha_1 I + \alpha_2 I^2} - (d + \delta_a)S > -(d + \delta_a)S$$

$$S(t) \geq S(0)e^{-(d+\delta_a)t} \geq 0.$$

Using the same patterns it can be expressed as,

$$I(t) \geq \frac{I(0)}{e^{(\lambda_i+\theta_i+d)t}}, A(t) \geq \frac{A(0)}{e^{(d+\sigma_a)t}} \text{ and } T(t) \geq \frac{T(0)}{e(d+\alpha)t}.$$

4. Equilibria and Basic Reproduction number

4.1 Equilibrium's

The system (1) reflect the below mention equilibrium points:
(i) The equilibrium point without any disease $\mathbf{E_0} = \left(\frac{R_s}{d+\delta_a}, 0, 0, 0\right)$ **and**
(ii) The equilibrium point with disease $\mathbf{E}^* = (\mathbf{S}^*, \mathbf{I}^*, \mathbf{A}^*, \mathbf{T}^*)$, where

$$\mathbf{S}^* = \frac{R_s(1 + \alpha_1 I^* + \alpha_2 I^{*2})}{\beta I^* + (d + \delta_a)(1 + \alpha_1 I^* + \alpha_2 I^{*2})},$$

$$\mathbf{A}^* = \frac{\theta_i I^*}{(d + \sigma_a)} \text{ and } \mathbf{T}^* = \frac{\lambda_i(d + \delta_a)I^* + \sigma_a \theta_i I^*}{(d + \alpha)(d + \sigma_a)} \textit{ and}$$

$\mathbf{I}^*$ is satisfies the equation

$$p_1 I^{*2} + q_1 I^* + r_1 = 0, \text{ where}$$

p_1=α_2>0,q_1=$\alpha_1 + \frac{\beta}{(d+\delta_a)}$>0 and$r_1 = (1 - R_0)$. It is obvious that when $\mathbf{R_0}$<1, then r_1>0 and the above equation has no positive roots. On the other hand when $\mathbf{R_0}$>1, then r_1<0 and the equation has at least one positive root, Therefore it is proven that when $\mathbf{R_0}$>1, then there exists a positive equilibrium point with the disease.

4.2 Basic Reproduction Number

The proposed model either consists of an equilibrium point with or without disease it depends on the basic reproduction number $\mathbf{R_0}$. With the help of the next-generation matrix technique find the value of $\mathbf{R_0}$. Here suppose the infection and non infection terms by the non-negative, nonsingular matrix, m_i and m_{ni} respectively.

$$\mathbf{m_i} = \begin{pmatrix} \frac{\beta SI}{1 + \alpha_1 I + \alpha_2 I^2} \\ 0 \end{pmatrix} \textbf{ and}$$

$$\mathbf{m_{ni}} = \begin{pmatrix} (\lambda_i + \theta_i + d)I \\ -\theta_i I + (d + \sigma_a)A \end{pmatrix}.$$

Now,
$\mathbf{M_i}$ = Jacobian of 'm_i' at disease-free equilibrium point=$\begin{pmatrix} \frac{R_s\beta}{d+\delta_a} & 0 \\ 0 & 0 \end{pmatrix}$
$\mathbf{M_{ni}}$= Jacobian of 'm_{ni}' at the disease-free equilibrium point=$\begin{pmatrix} (\lambda_i + \theta_i + d) & 0 \\ -\theta_i & (d + \sigma_a) \end{pmatrix}$

$$R_0 = \rho M_i M_{ni}^{-1} = \text{Spectral radious}$$
$$= \frac{R_s\beta}{(d + \delta_a)(\lambda_i + \theta_i + d)}$$

5. Stability Analysis

5.1 Local Stability

Here analysis analytically, the local stability of E_0 and E^* of the model (1) in the following theorems.

Theorem 5.1.1. The equilibrium point without disease E_0 is locally asymptotically stable if, R_0< 1, and unstable while R_0>1.

Proof: The Jacobian matrix J_0 at E_0 is given by

$$J_0 = \begin{pmatrix} -(d + \delta_a) & -\frac{R_s\beta}{(d + \delta_a)} & 0 & 0 \\ 0 & \frac{R_s\beta}{(d + \delta_a)} - (\lambda_i + \theta_i + d) & 0 & 0 \\ 0 & \theta_i & -(d + \sigma_a) & 0 \\ 0 & \lambda_i & \sigma_a & -(d + \alpha) \end{pmatrix}$$

The characteristics equation associated with$J_0$is mentioned below
[-(d + α) − e_{0i}][−(d + σ_a) − e_{0i}[−(d + δ_a) − *e0i*Rsβd+δa−λi+θi+d−*e0i*=*0*

The eigenvalues associated to above characteristic equation is

$$e_{01} = -(d + \alpha) < 0, e_{02} = -(d + \sigma_a) < 0, e_{03} = -(d + \delta_a) < 0 \textit{ and } e_{04} = \frac{R_s\beta}{(d + \delta_a)} - (\lambda_i + \theta_i + d) = (\lambda_i + \theta_i + d)(R_0 - 1) < 0 \textit{ for } R_0 < 1.$$

Therefore it is clear the characteristic equation has four negative eigenvalues while, $R_0 < 1$. So the equilibrium point without disease E_0 is locally asymptotically stable when

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$R_0 < 1$. When $R_0 > 1$, then e_{04}>0 and the characteristic equation has one positive and three negative eigenvalues, so a saddle unstable equilibrium exists.

Theorem 5.1.2 For $R_0 > 1$, the endemic equilibrium point E^* is locally asymptotically stable.

Proof: The Jacobian matrix J_e^* about the endemic equilibrium point E^* is given by,

$$J_e^* = \begin{pmatrix} -p-(d+\delta_a) & -q+\alpha_2 r & 0 & 0 \\ p & q-\alpha_2 r-(\lambda_i+\theta_i+d) & 0 & 0 \\ 0 & \theta_i & -(d+\sigma_a) & 0 \\ 0 & \lambda_i & \sigma_a & -(d+\alpha) \end{pmatrix}, \text{ where}$$

$$p = \frac{\beta I^*}{1+\alpha_1 I^* + \alpha_2 I^{*2}}, \quad q = \frac{\beta S^*}{(1+\alpha_1 I^* + \alpha_2 I^{*2})^2} \text{ and } r = \frac{\beta S^* I^{*2}}{(1+\alpha_1 I^* + \alpha_2 I^{*2})^2}$$

The two eigenvalues of the above matrix are e_1^*=-(d+α), e_2^*=-(d+σ_a) and another two eigenvalues of the below-mentioned characteristics equation related to the matrixJ_e^*.

The characteristics equation of J* is

$$e^{*2} + \rho e^* + \rho_1 = 0, \quad \text{where}$$
$$\rho = p - q + \alpha_2 r + (d+\delta_a) + (\lambda_i + \theta_i + d)$$
$$\rho_1 = p(\lambda_i + \theta_i + d) + (\lambda_i + \theta_i + d)(d+\delta_a) + (d+\delta_a)(\alpha_2 - r)$$

Now $\rho > 0$ and ρ_1>0 because p>q and q<r therefore the above quadratic equation has two roots with negative real parts. Thus the endemic equilibrium point E^* is locally asymptotically stable when $R_0 > 1$.

6. Sensitivity Analysis

This segment tries to find a few parameters of the model (1) which is more sensible to prevent this type of disease transmission. The below table displays the sensible values of all parameters on R_0. From the various studies on epidemiology, the different types of diseases can be spread or not completely trust the value of the basic reproductive number. So, it is too much necessary to detect a few sensible parameters. Calculate the sensible parameters and their values with the assistance of the formula [20].

$$S_{R_0}^{s_i} = \frac{\partial R_0}{\partial s_i} \times \frac{s_i}{R_0}$$

All the non-negative values of $S_{R_0}^{s_i}$ in Table 1 suggest that if enhance (decrease) the sensible parameters value which is associated with R_0, then R_0its value will also enhance (decrease). Also, the negative values of $S_{R_0}^{s_i}$in Table 1 suggest that if enhance (decrease) the sensible parameters its value which is associated withR_0, then R_0its value will also decrease (enhance). From this analysis, it can be ensured that the most desirable sensible, and manageable parameter is awareness and treatment.

Table 1: Sensible values of all parameters on R_0

Parameter s_i	R_s	β	d	δ_a	λ_i	θ_i
Value	0.78	0.1	0.04	0.01	0.7	0.02
$S_{R_0}^{s_i}$	1	1	-0.85	-0.20	-0.92	-0.03

7. Numerical Explanations

Now established the analytical outcomes with the help of the numerical technique of the model (1). Figure 1 shows that E_0 is locally asymptotically stable while R_s=0.35, R_0=0.9211<1 and other parameter values are taken from Table 2. Figure 2 displays that the disease-related equilibrium point E^* is stable asymptotically when R_0=2.0526 and all parameter values are taken from Table 2. Figure 3 indicates that the sensitivity analysis results of all parameters of the model (1) on R_0. Lastly, Figures 4(a) and 4(b) indicate that if the value of treatment and awareness rate is certain then the value of R_0will decrease.

Table 2: A set of parametric values

Parameter	Definition	Default Value
R_s	Increase rate of susceptible class	0.78
β	Is the Contact rate between susceptible and HIV infected class	0.1
α_1	A positive constant	0.01
α_2	A positive constant Mortality rate of adults class	0.02
d	Death rate of adult class	0.04
δ_a	Is the Awareness rate of susceptible class	0.01
λ_i	The rate at which the infected class is recruited into the treatment class	0.7
θ_i	The rate at which the infected class is recruited into the AIDS class	0.02
σ_a	Is the treatment rate of AIDS class	0.1
Α	Is the disease induced death rate due to treatment	0.6

8. Conclusion

In this paper, consider and discuss an HIV/AIDS model with a non-linear saturated incidence rate and study the effect of awareness and treatment. Calculate a basic reproduction numberR_0. Analytical results show that the system (1) has no disease when R_0<1 and it persists when R_0>1. The system (1) consists of two nonnegative equilibrium points i.e. without disease and with a disease which is denoted by E_0 and E^*. From the analytical results show that both the equilibrium points are locally asymptotically stable and it also justified numerically. Also, discuss the sensitivity analysis of some parameters which is involved inR_0. After analyzing found some sensitive parameters that are more sensible to control the changes of value R_0. If increase the value of awareness and treatment rate then the value of R_0will also decrease and go below one. It is also found that if we increase the treatment rate then the intensity of the epidemic decreases. Studies results say that increased awareness i.e. use of condoms before sex, media cam-panning, various awareness programs, etc. then can slow down the disease transmission. Finally, concludes that if can treat the infected persons at an early stage regularly then disease transmission can slow down and enhance the lives of all infected individuals. Even though regular basis treatment is so expensive and it is not available so many countries of the World.

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Appendices:

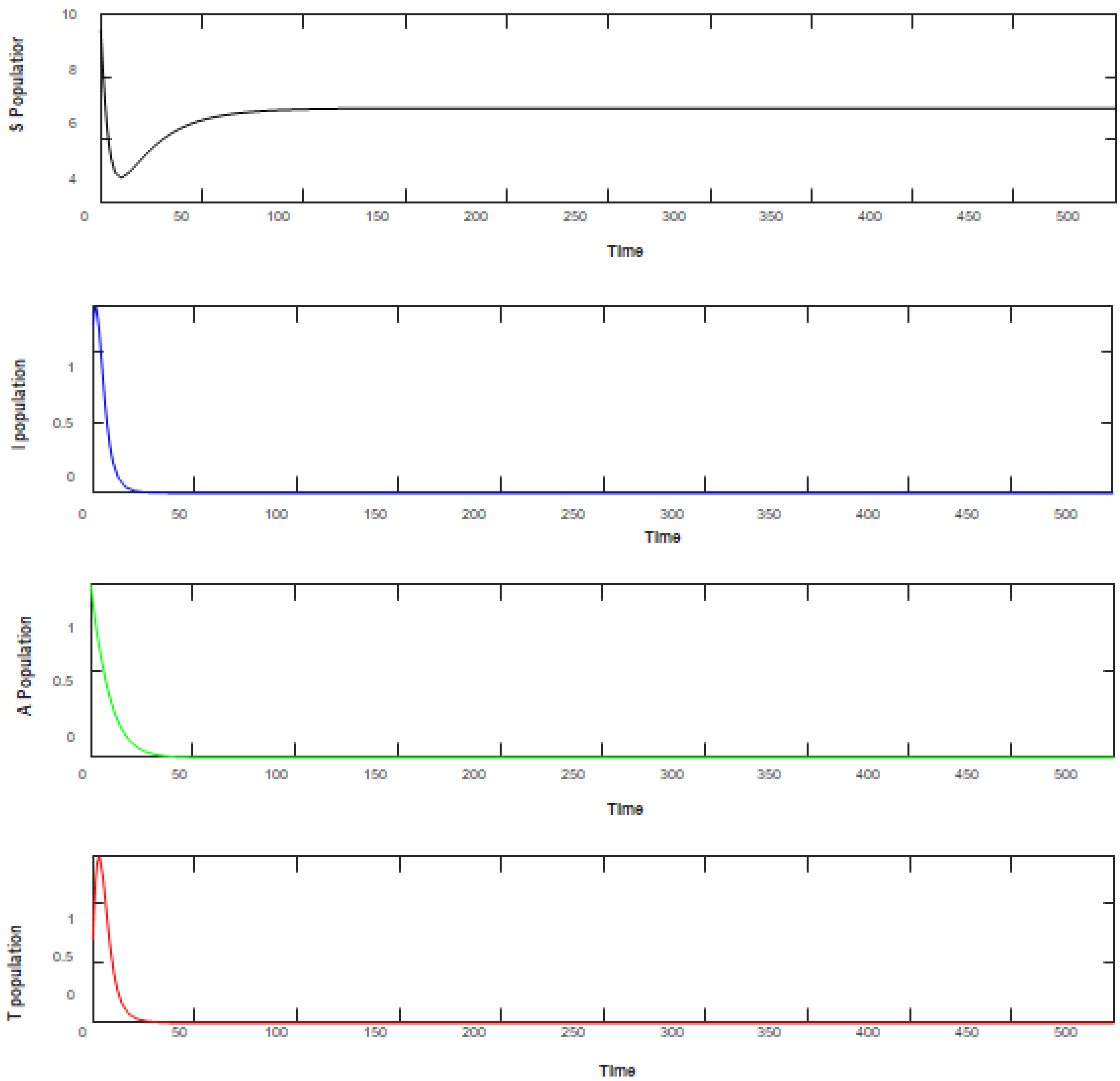
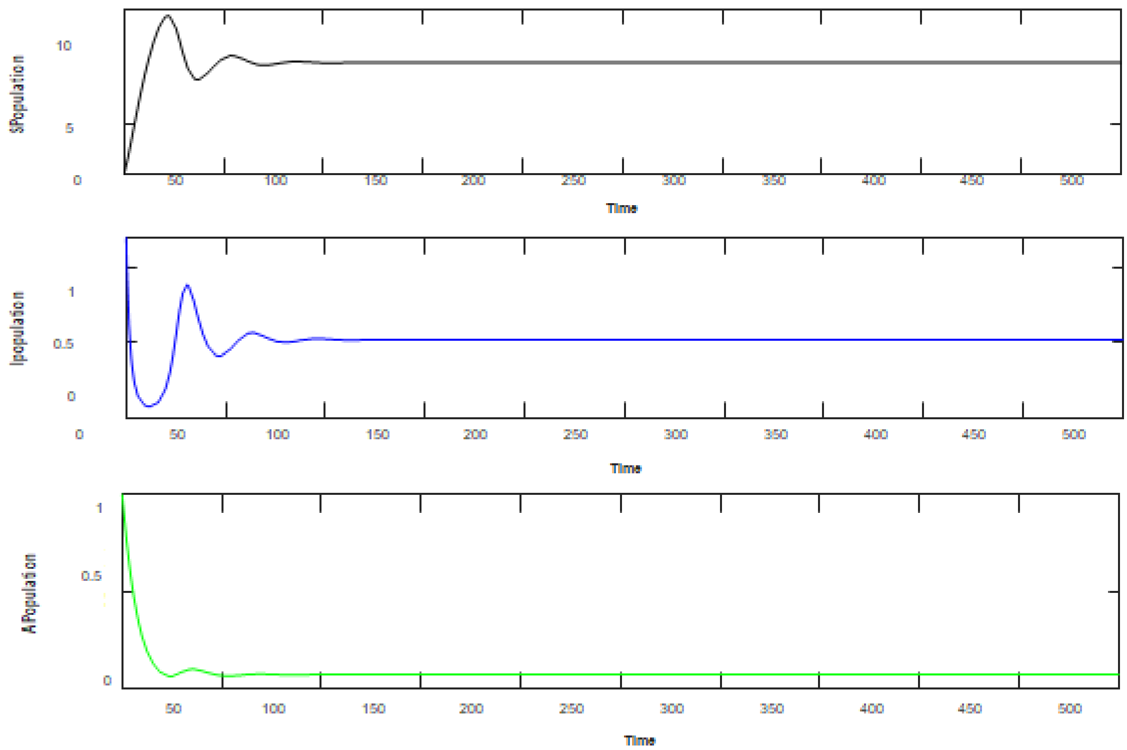


Figure 1: E_0 is asymptotically stable for while R_0<1.



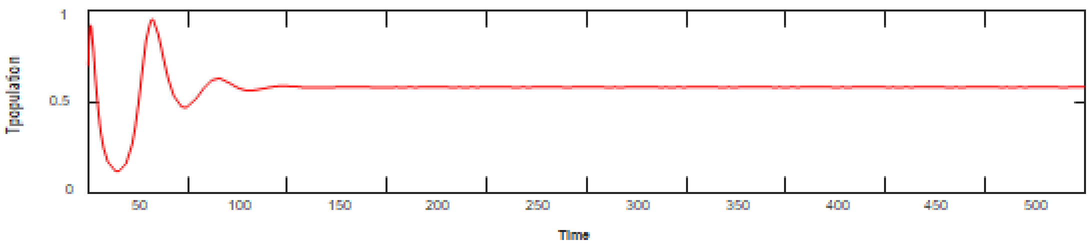


Figure 2: The equilibrium point E^* is locally asymptotically stable $R_0>1$

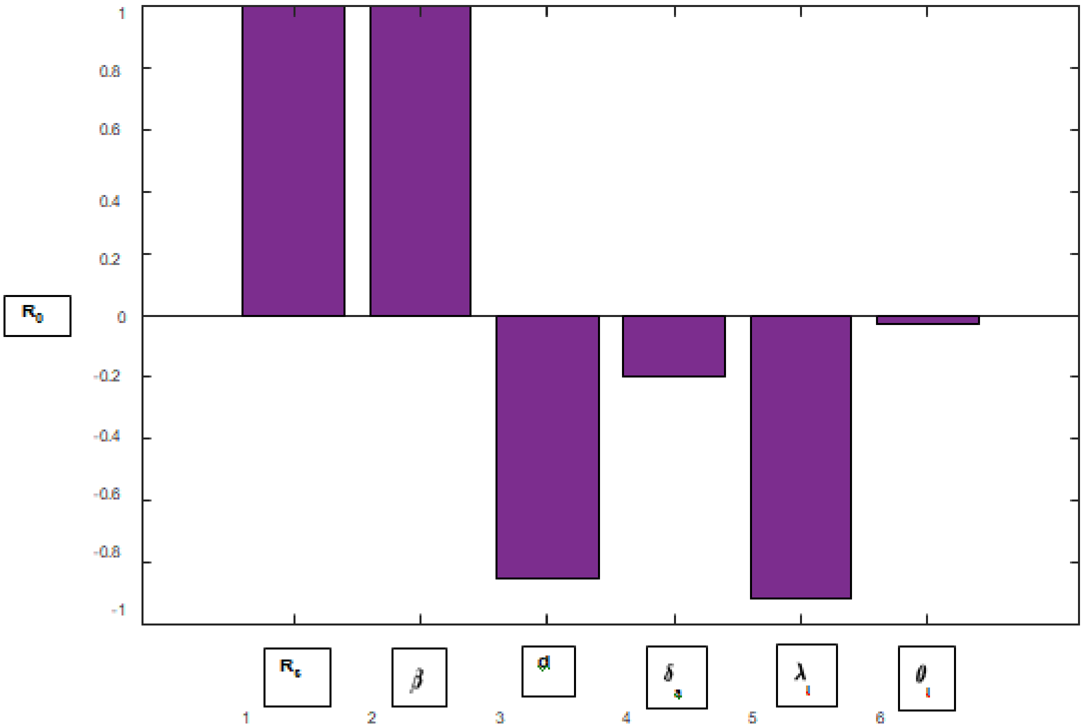


Figure 3: Sensitivity analysis values of all the parameters of the system (1) on R_0.

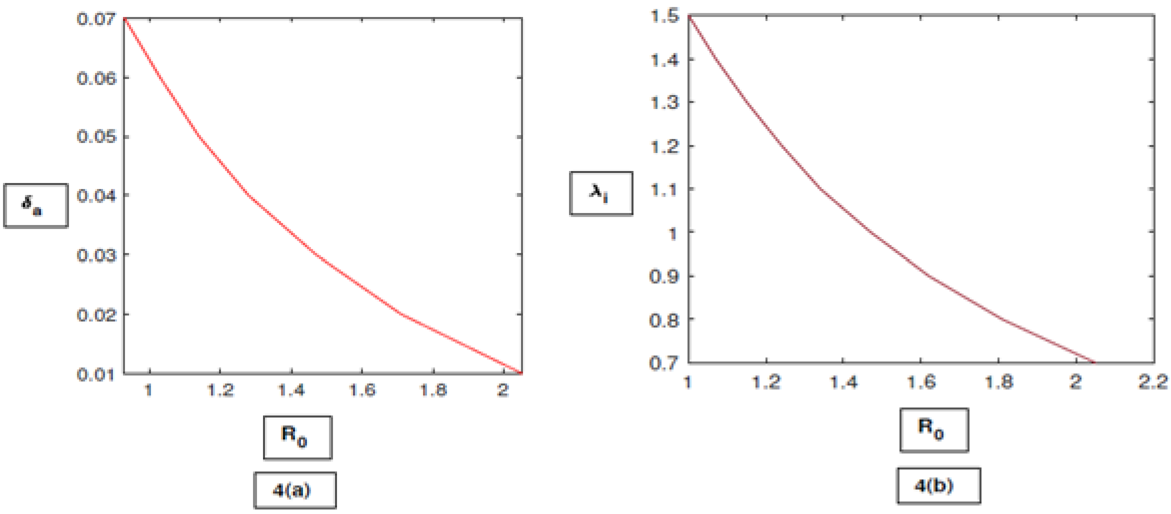


Figure 4: Figure 4 (a) displays that if the value of δ_a increases the nR_0 its value decreases Figure 4(b) indicates that if λ_i increases, then R_0's value will decrease.

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