Application of GeoGebra in Teaching Open Set and Interior Set in Metric Spaces

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Abstract: *Topology of metric spaces is a standard and important course at the college level mathematics studied at University of Mumbai. As the course is abstract, students feel it as a difficult course and hence tend to develop a fear. In this paper, we will see how to use the open-source software GeoGebra to teach the concepts of this course effectively. We will concentrate on the topics of open set and interior set. GeoGebra applets are created for various examples in different metric spaces to teach these concepts efficiently.*

Keywords: GeoGebra, metric space, open set, interior set

1. Introduction

GeoGebra is open-source software for mathematics teaching and learning that offers geometry, algebra and calculus features in a fully connected and easy-to-use software environment. It is available free of charge and used by thousands of students and teachers around the world in classrooms and at home (Hohenwarter et al., 2008). GeoGebra is a dynamic mathematics software for teaching and learning mathematics from middle school through college level. GeoGebra was created by Markus Hohenwarter to help students gain a better understanding of mathematics (Hohenwarter et al., 2007). GeoGebra is highly useful software both for teachers as well as students. Teachers can use GeoGebra to facilitate interactive and engaging lessons. The software allows for real-time demonstrations, enabling teachers to illustrate concepts dynamically and adapt to students' needs. GeoGebra provides a platform for students to experiment with different scenarios and visualize solutions, leading to a deeper understanding of the underlying concepts. Students can interact with mathematical objects, manipulate parameters, and observe how changes affect outcomes, enhancing their comprehension and retention of mathematical concepts.

GeoGebra can be a valuable tool for exploring concepts related to metric spaces. We can use GeoGebra to create visual representations of metric spaces. For example, we can plot points in a plane or in three-dimensional space and use GeoGebra's tools to measure distances between them or explore properties of metric spaces such as triangle inequality and so on. In this paper, we are using GeoGebra for creating interactive applets to learn the concepts of open set and interior set in metric spaces.

Open Ball:
Definition: Let (X, d) be a metric space and $p \in X$. An open ball with centre p and radius $r > 0$, denoted by $B(p, r)$, is defined as follows (Kumaresan, 2011):
$$B(p, r) = \{x \in X : d(x, p) < r\}$$

Example 1:
Consider $\mathbb{R}^2$ with Euclidean metric d_2 given by $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Let $p = (a, b) \in \mathbb{R}^2$ and $r > 0$. Then open ball with centre p and radius r is given by

$$\begin{aligned} B(p, r) &= \{(x, y) \in \mathbb{R}^2 : d_2((x, y), (a, b)) < r\} \\ &= \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < r^2\} \end{aligned}$$

GeoGebra applet https://www.geogebra.org/m/b7kx7ye7 was created for effective explanation of this open ball (Patne, 2024).
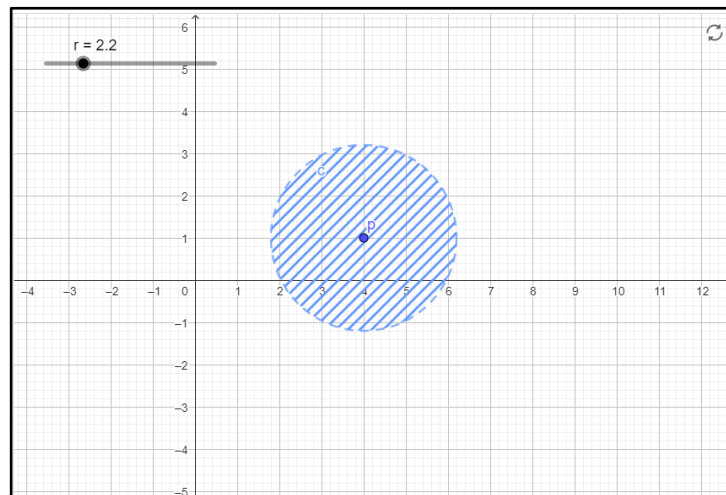
Figure 1: Open Ball in $(\mathbb{R}^2, d_2)$

Example 2:
Consider $\mathbb{R}^2$ with sum metric given by $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Let $p = (a, b) \in \mathbb{R}^2$ and $r > 0$. Then open ball with centre p and radius r is given by

$$B(p,r) = \{(x,y) \in \mathbb{R}^2 : d_1\big((x,y),(a,b)\big) < r\}$$
$$= \{(x,y) \in \mathbb{R}^2 : |x - a| + |y - b| < r\}$$

GeoGebra applet https://www.geogebra.org/m/khxfwcer can be used for demonstrating the open ball.
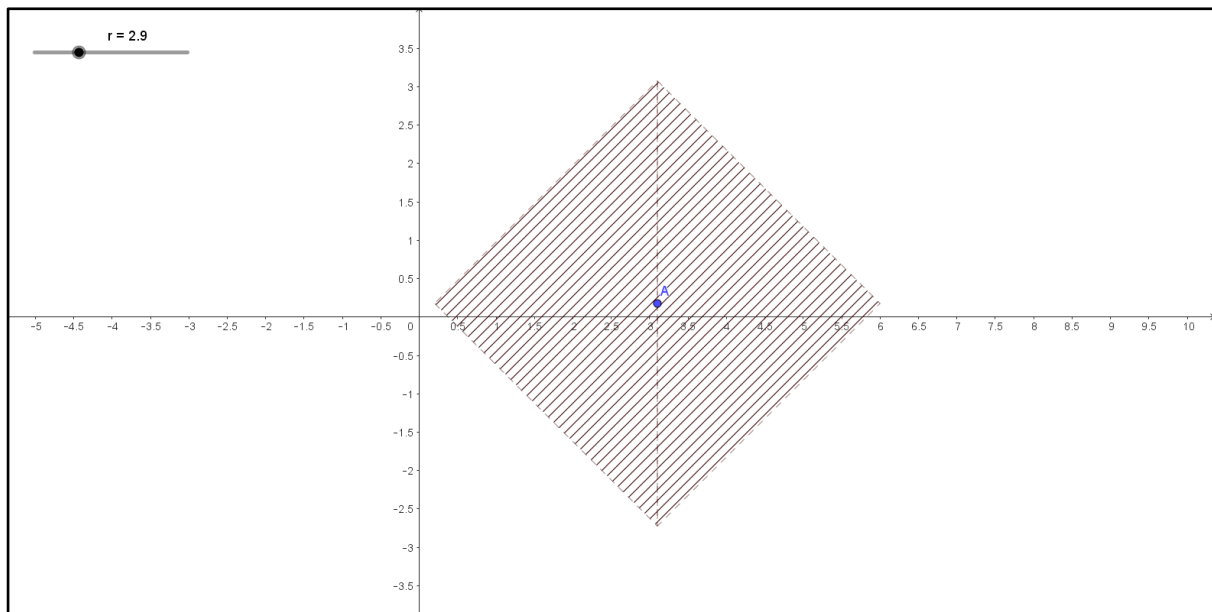

Figure 2: Open Ball in $(\mathbb{R}^2, d_1)$

Example 3:
Consider the set of all continuous real valued functions defined on interval $[0,1]$ denoted by $C[0,1]$ with sup norm $d(f,g) = \sup\{|f(t) - g(t)| : t \in [0,1]\}$. GeoGebra applet https://www.geogebra.org/m/px5vkjdt is created for the following example in the metric space $C[0,1]$:

Let $f(x) = x^3 + 5x - 1$ for $x \in [0,1]$. Then $f \in C[0,1]$. Open ball with centre f will be given by

$$B(f,r) = \{g \in C[0,1] : d(f,g) < r\}$$
$$= \{g \in C[0,1] : \sup\{|f(t) - g(t)| : t \in [0,1]\} < r\}$$
$$= \{g \in C[0,1] : |f(t) - g(t)| < r \text{ for all } t \in [0,1]\}$$
$$= \{g \in C[0,1] : f(t) - r < g(t) < f(t) + r \text{ for all } t \in [0,1]\}$$
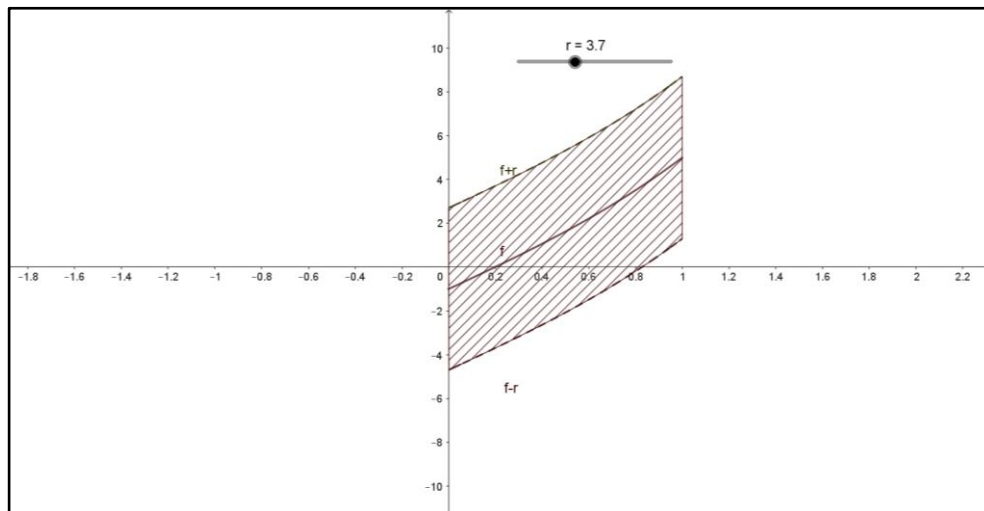
Figure 3: Open Ball in C[0,1] with sup norm

Open Set:

Definition: Let (X, d) be a metric space and G be any subset of X. We say that G is an open subset of X if for all $p \in G$, there exits $r > 0$ such that $B(p, r) \subseteq G$.

Example 1:

Consider $\mathbb{R}^2$ with Euclidean metric d_2 given by $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Let $G = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Then it is a standard proof in the course of topology of metric space to prove that G is an open subset of $(\mathbb{R}^2, d_2)$. To support the understanding of students, we can use the applet https://www.geogebra.org/m/wcayxzqs created in GeoGebra. The applet will help students to check whether the condition of open set is satisfied in this example. As we change $p \in G$, students are encouraged to find the value of radius which will satisfy the condition $B(p, r) \subseteq G$.
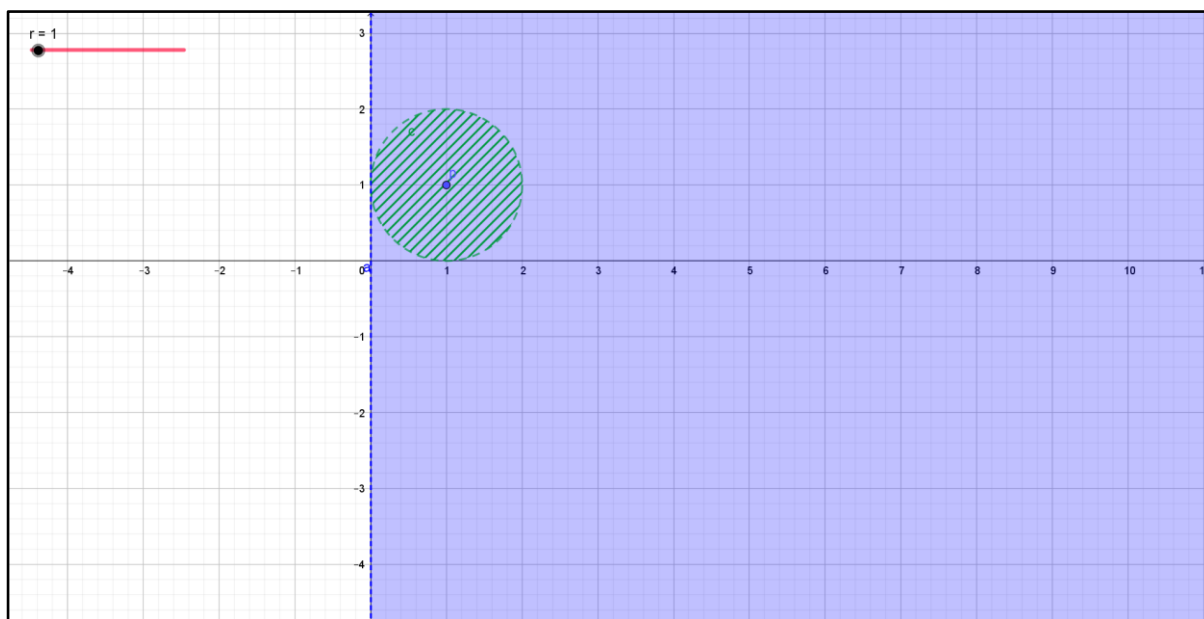

Figure 4: Open set in $(\mathbb{R}^2, d_2)$

Hence using the applet, it becomes easy to conclude that for $p = (a, b) \in G$, we take $r = a$ such that $B(p, r) \subseteq G$.

Example 2:

Consider $\mathbb{R}^2$ with Euclidean metric d_2 given by $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Let $G = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$. GeoGebra applet created for this set can be found at https://www.geogebra.org/m/fyncfhwz. Here one can observe that the open ball around any point $p \in G$ is not contained in G for any value of r. The applet helps students to visualize that however small radius is taken, open ball cannot be contained in G. So the applet helps students to observe that G is not an open set.
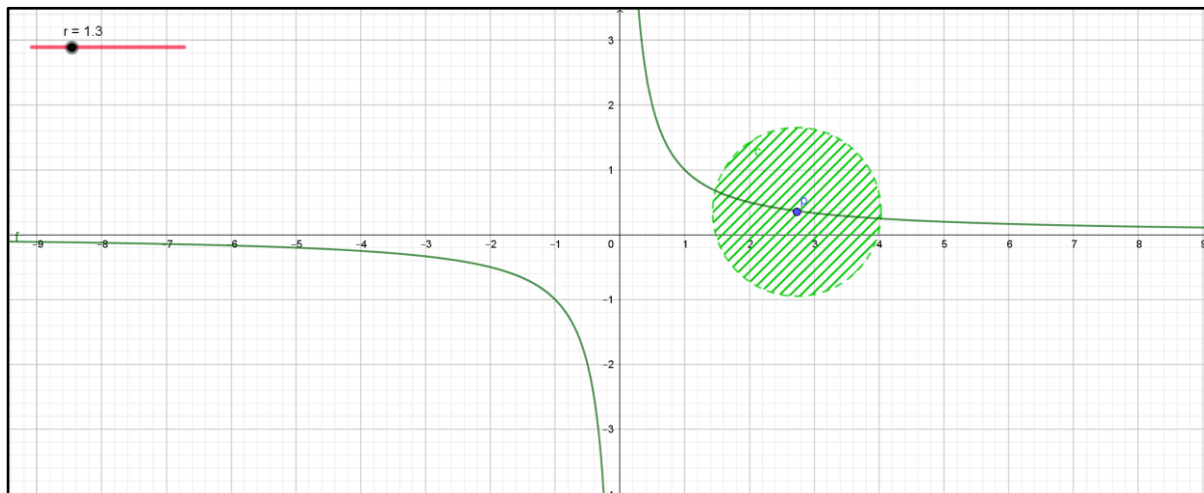
Figure 5: Example of a non-open set

Interior points:

Let (X, d) be a metric space and A be any subset of X. We say that $p \in X$ is an interior point of A if there exits $r > 0$ such that $B(p, r) \subseteq A$.

Interior set:

Let (X, d) be a metric space and A be any subset of X. The set of collection of all interior points of set A is called interior of A and is denoted by A^o.

Example 1:

Consider $\mathbb{R}^2$ with sum metric given by $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Let $p = (a, b) \in \mathbb{R}^2$ and $r > 0$. Then open ball with centre p and radius r is given by

$$B(p, r) = \{(x, y) \in \mathbb{R}^2 : d_1((x, y), (a, b)) < r\}$$
$$= \{(x, y) \in \mathbb{R}^2 : |x - a| + |y - b| < r\}$$

Consider the problem of finding interior set of $A = \{(x, y) : x - y < 0\} \subseteq \mathbb{R}^2$ with respect to metric d_1.

GeoGebra applet https://www.geogebra.org/m/umwjsnkd can be used effectively to find the interior points of above set. In this applet, students can change the point p all over $\mathbb{R}^2$ by dragging the point. They can change the value of radius r using the slider. To check whether the condition of interior point is satisfied by $p = (a, b)$, students need to consider three cases: $a < b, a = b$ and $a > b$.

We can create a dynamic worksheet in Google classroom or any LMS using above applet. In this, ask students to select one point for each of the above cases and change the radius to check whether $B(p, r) \subseteq A$.

Let us see an example in this applet for each of the above cases.

Case i): $a < b$

For this case, let us take $p = (-1, 3)$. As shown in the Figure below, the condition is satisfied for $r = 2$. That is, $B(p, 2) \subseteq A$. Hence $p = (-1, 3)$ is an interior point of A. Students can verify that the condition is satisfied for any p in this case.
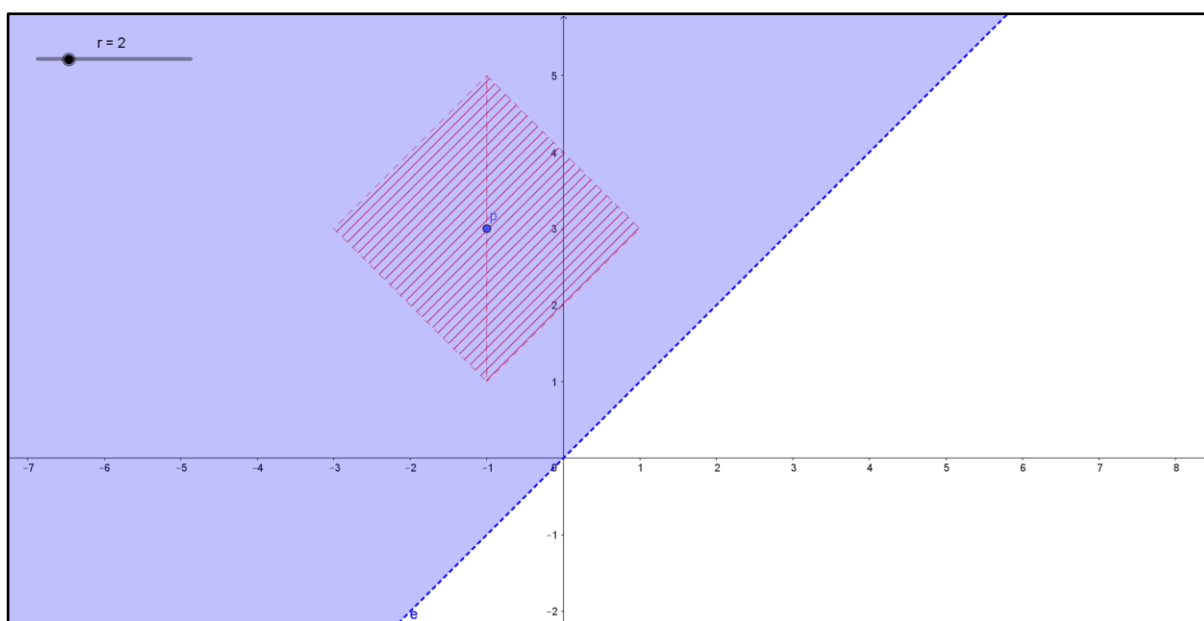

Figure 6: p=(-1,3) as interior point of A

In the figure below, again a different example is taken for case 1. The point $p = (0.5,1)$ is an interior point of A since $B(p, 0.5) \subseteq A$. This example can be used to emphasis among the students that although the point is very near to the line $x = y$, the condition is satisfied. We need to take r as small as possible.
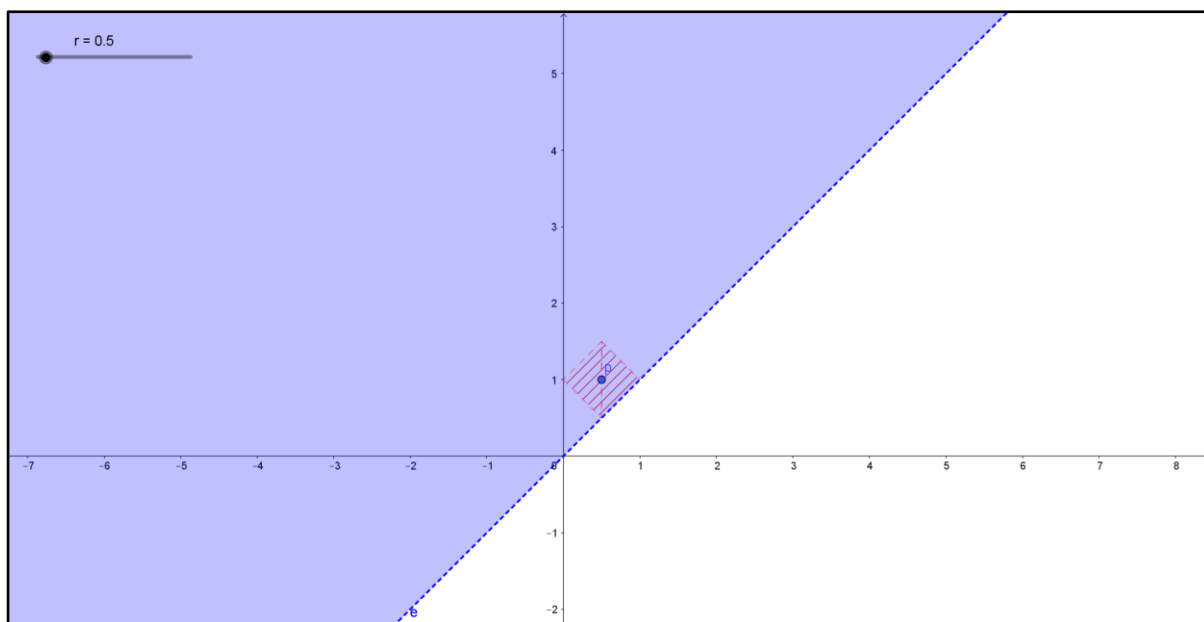

Figure 7: p= (0.5,1) as interior point of A

Case ii): $a = b$

Here all the points lie on the line $x = y$. In the figure below, for $p = (1,1)$, students can observe that the open ball is not contained in the set A for any value of r. We can take r as small as possible, still the condition is not satisfied. Hence condition is not satisfied in this case. So p is not an interior point of set A.
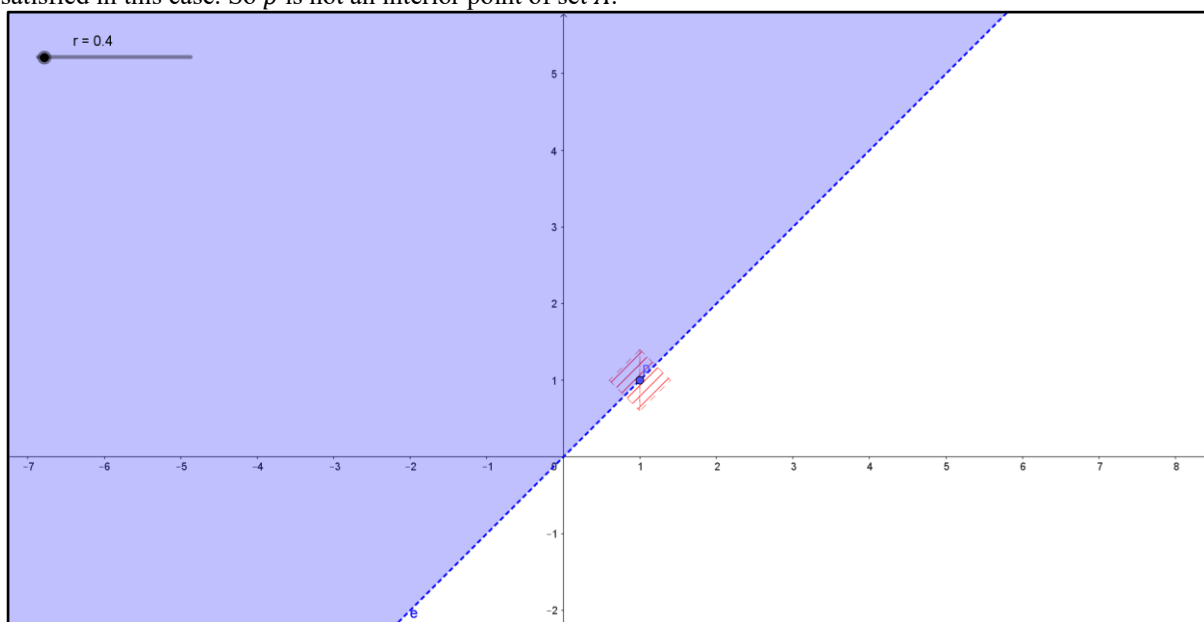

Figure 8: p= (1,1) as not an interior point of A

Case iii) $a > b$

Same as in case ii), students should be encouraged to change the value of r and note that the open ball is not contained in set A for any value of r. So condition is not satisfied in this case also.
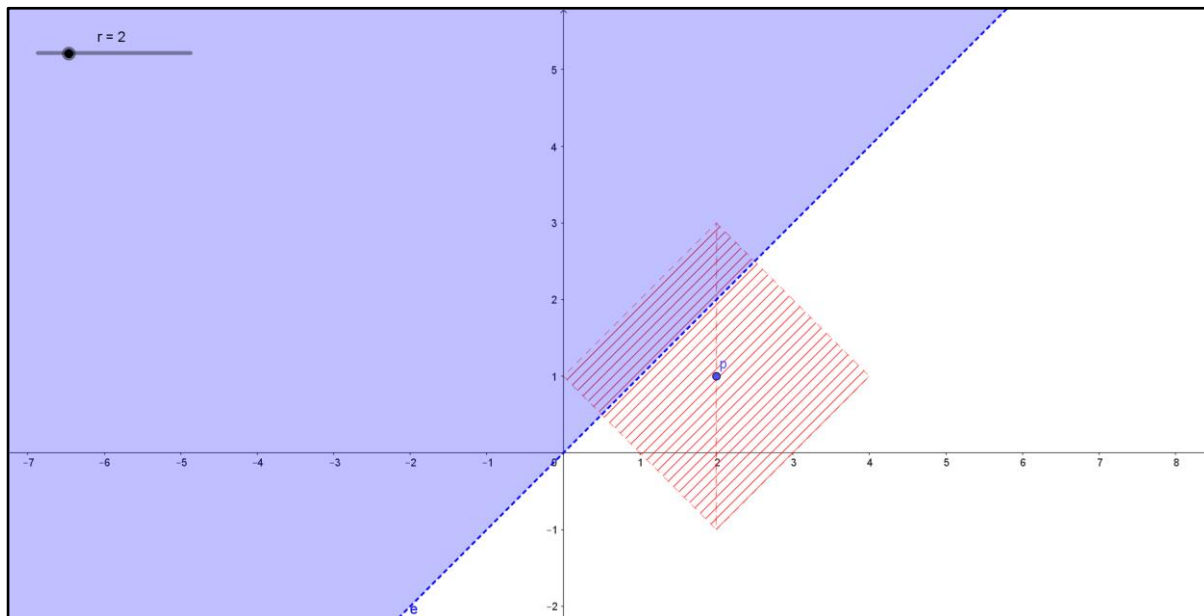
Figure 9: p=(2,1) as not an interior point of A

After observing all the above cases, students should reach the conclusion that $A^o = A$ in this example.

Example 2:
Consider $\mathbb{R}^2$ with Euclidean metric d_2 given by $d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Now let us find interior of the set $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ in $(\mathbb{R}^2, d_2)$.

In this example, students need to consider points $p = (x,y)$ in three cases as follows:
$$x^2 + y^2 < 1, \ x^2 + y^2 = 1 \text{ and } x^2 + y^2 > 1$$

GeoGebra applet https://www.geogebra.org/m/qkpaknnt is used for above example. Following figures depict the above three cases.

Case 1: $x^2 + y^2 < 1$

In this case, students can observe that the condition of interior point is satisfied if we keep radius $r < 1 - d(p,0) = 1 - x^2 - y^2$. This radius is shown in figure below using red line segment. So all points in this case are interior points of B.
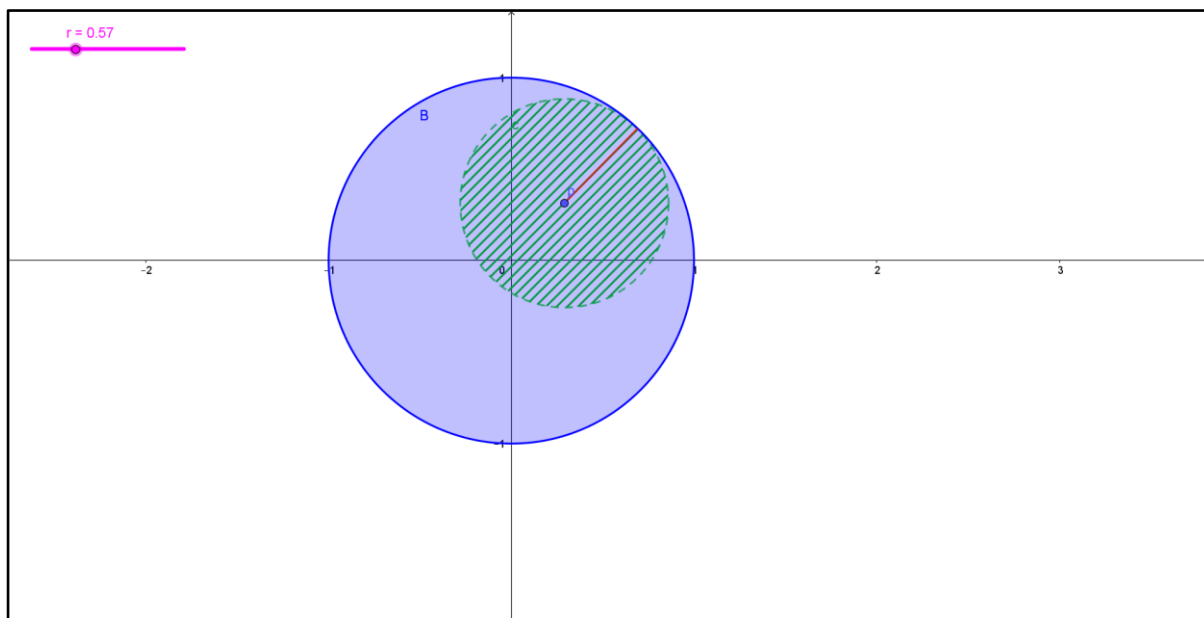

Figure 10: Case 1

Case 2: $x^2 + y^2 = 1$ and Case 3: $x^2 + y^2 > 1$

In both these cases, the applet makes it very easy for students to figure out that the condition is not satisfied as shown in figures below.
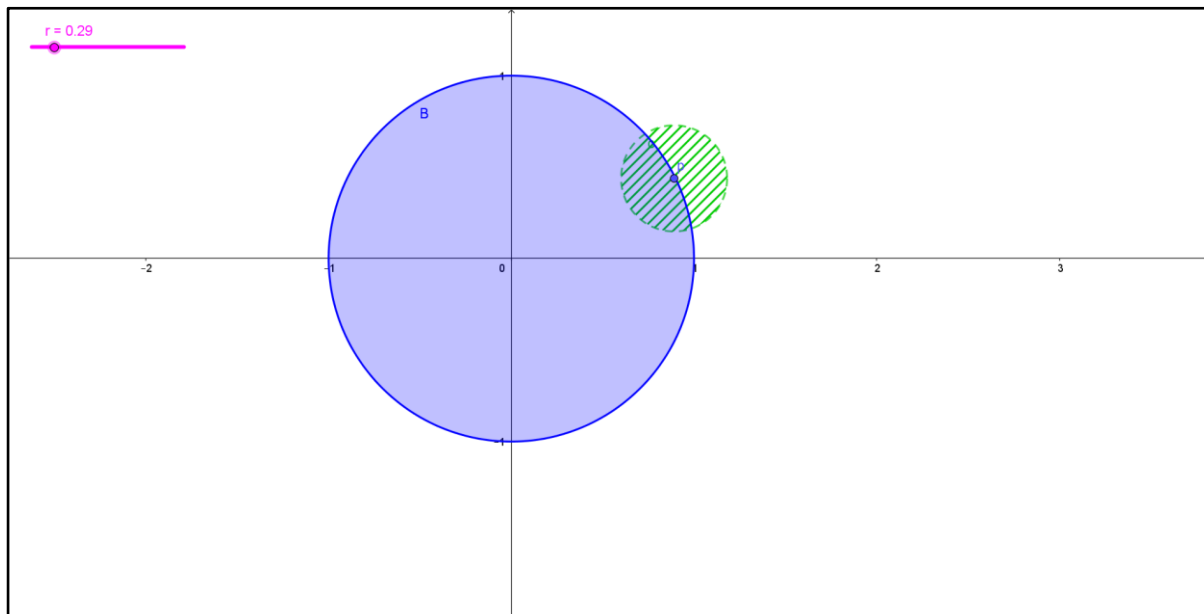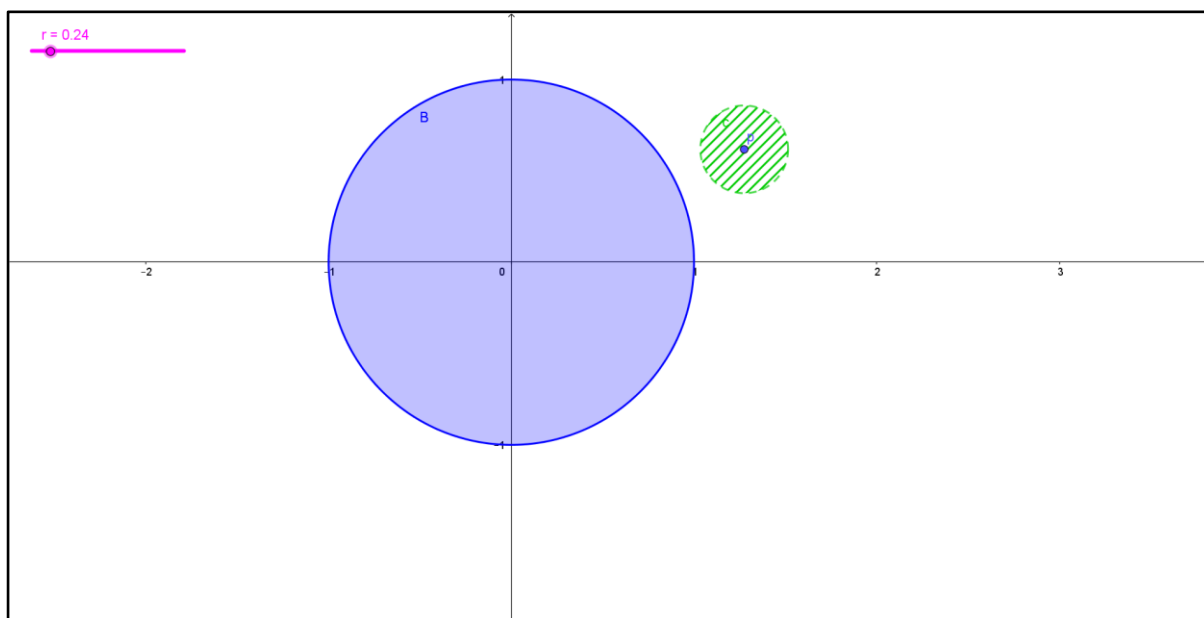
Figure 11: Case 2

Figure 12: Case 3

Hence students can conclude that $B^o = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

2. Conclusion

Topology of Metric Spaces can be a challenging course to undergraduate mathematics students due to its abstract nature and the intricacies involved in studying the properties of sets and functions with respect to metrics. However, with the use of GeoGebra, we can break down the concepts and work through examples to teach this course with better understanding. GeoGebra applets created in this paper will help students visualize the concepts of open set and interior set and gain a better intuitive understanding.

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