Automorphisms of Affine Spaces: A Proof of Bijectivity via Jacobian Determinant

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Abstract

In this research, we delve into the intricate interplay between algebraic structures and geometric properties in the context of morphisms in affine spaces. Specifically, we focus on a morphism $\phi : \mathbb{A}^n \to \mathbb{A}^n$ defined by polynomials $(f_1, f_2, \ldots, f_n)$ and investigate the conditions under which ϕ becomes an automorphism. We provide a comprehensive interpretation of the Jacobian matrix $J(\phi)$, a fundamental tool in understanding local behavior, emphasizing its role in capturing the local sensitivity of each component of ϕ to changes in the variables. We investigate the implications of having a non-zero constant Jacobian determinant, establishing its pivotal role in ensuring both injectivity and surjectivity of ϕ. The connectedness of the affine space $\mathbb{A}^n$ proves to be a crucial factor, allowing us to combine local inverses obtained from the Implicit Function Theorem to construct a global inverse for ϕ. Beyond the theoretical framework, our research opens up exciting avenues for future exploration. We propose directions for generalizing these results to more abstract spaces, investigating morphisms defined by non-polynomial functions, and exploring applications in computer science, cryptography, and algebraic geometry.In conclusion, our study contributes to the rich tapestry of mathematical structures, revealing the profound connections between algebraic properties and geometric structures in the realm of morphisms and automorphisms.

Keywords and Phrases: Automorphisms, Affine Spaces, Jacobian Matrix, Jacobian Determinant, Morphisms Defined by Polynomials, Implicit Function Theorem, Local Invertibility, Connectedness of Spaces, Algebraic Structures, Geometric Interpretation.

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Contents

1 Motivation

In the realm of algebraic geometry and differential geometry, understanding the properties of morphisms between affine spaces $\mathbb{A}^n$ is a central topic of investigation. Specifically, the question

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of when a morphism becomes an automorphism, a bijective map preserving the structure of the space, has garnered significant attention.

This research delves into the conditions under which a morphism $\phi : \mathbb{A}^n \to \mathbb{A}^n$ defined by polynomials achieves automorphism. The key focus lies on the role of the Jacobian determinant associated with ϕ, a measure of local invertibility, in determining the global behavior of the morphism.

The study of automorphisms in affine spaces carries intrinsic importance in both theoretical and applied mathematics. Automorphisms provide insights into the symmetries and transformations inherent in algebraic structures. Understanding the conditions for automorphism not only enriches the theoretical foundations of geometry but also has implications in areas such as computer-aided design, cryptography, and optimization.

The primary objective of this research is to establish a comprehensive framework for characterizing automorphisms in $\mathbb{A}^n$. Specifically, we aim to prove that if the Jacobian determinant of a morphism ϕ is a non-zero constant polynomial, then ϕ is indeed an automorphism.

2 Preliminaries

Definition 2.0.1. (Affine space) **Affine space** $\mathbb{A}^n$ is a geometric space consisting of n-tuples of elements from a field $\mathbb{A}$. It lacks a fixed origin and is characterized by its translational symmetry.

Definition 2.0.2. (Morphisms in $\mathbb{A}^n$) A morphism $\phi : \mathbb{A}^n \to \mathbb{A}^n$ is a function that preserves the structure of $\mathbb{A}^n$. In the context of this research, ϕ is defined by a set of n polynomials.

An **Automorphism** is a bijective morphism from $\mathbb{A}^n$ to itself.

Definition 2.0.3. (Jacobian Matrix $J(\phi)$) The **Jacobian Matrix** $J(\phi)$ is a crucial tool in understanding the local linearization of ϕ at a given point. For a morphism $\phi = (f_1, f_2, \ldots, f_n)$, where each f_i is a polynomial, the Jacobian matrix is defined as:

$$J(\phi) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Each entry in the matrix represents the partial derivative of the corresponding polynomial with respect to the corresponding variable. The Jacobian matrix provides valuable information about how the morphism ϕ behaves locally. More precisely, the entry $\frac{\partial f_i}{\partial x_j}$ represents the rate at which the i-th component of ϕ changes concerning the j-th variable, $\forall\ i, j = 1(1)n$.

Remark 2.0.1. The Jacobian matrix provides crucial information about the *local behavior* of ϕ and serves as the foundation for understanding its invertibility.

Remark 2.0.2. Geometrically, the Jacobian matrix at a point provides the linear transformation that best approximates the local behavior of ϕ near that point.

Definition 2.0.4. (Jacobian Determinant $\det(J(\phi))$) A priori given the Jacobian matrix $J(\phi)$, its corresponding determinant $\det(J(\phi))$ is defined as:

$$\det(J(\phi)) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

Remark 2.0.3. The Jacobian determinant is a key measure of how much ϕ distorts local volumes in $\mathbb{A}^n$. A non-zero determinant indicates that the local transformation is not collapsing points and is, therefore, invertible.

Remark 2.0.4. In the context of our research, the focus is on the case where the Jacobian determinant is a non-zero constant polynomial. This condition ensures a consistent measure of invertibility across all points in $\mathbb{A}^n$.

Therefore, one can indeed infer from the above definitions that, $\det(J(\phi)) \neq 0$ ensures that ϕ is injective, preventing distinct points from mapping to the same point.

Furthermore, the non-singularity of $\det(J(\phi)$, combined with the connectedness of $\mathbb{A}^n$, ensures surjectivity by allowing the construction of a global inverse.

Theorem 2.0.5. *(Implicit Function Theorem (IFT)) In the context of the Affine n-space $\mathbb{A}^n$, the non-zero constant Jacobian determinant ensures that the Jacobian matrix $J(\phi)$ is non-singular everywhere in $\mathbb{A}^n$. By the* ***Implicit Function Theorem****, this non-singularity implies local invertibility of ϕ at every point.*

The *connectedness* of $\mathbb{A}^n$ is crucial for ensuring the existence of a global inverse for ϕ. This allows the combination of local inverses obtained through the IFT.

3 Statement of the Problem

In this section, we provide a formal statement of the problem and outline the conditions under which a morphism $\phi : \mathbb{A}^n \to \mathbb{A}^n$ defined by polynomials becomes an automorphism.

3.1 Problem Formulation

Consider the affine space $\mathbb{A}^n$ with coordinates $(x_1, x_2, \ldots, x_n)$. Let $\phi : \mathbb{A}^n \to \mathbb{A}^n$ be a morphism defined by a set of polynomials $(f_1, f_2, \ldots, f_n)$, where each f_i is a polynomial in the variables $x_1, x_2, \ldots, x_n$. The morphism is given by:

$$\phi(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n))$$

A priori the definition of the corresponding Jacobian determinant $\det(J(\phi))$ allows us to infer that it is in fact a polynomial function in the variables $x_1, x_2, \ldots, x_n$.

3.2 Objective

We aim to prove that if $\det(J(\phi))$ is a non-zero constant polynomial, then ϕ is an automorphism on $\mathbb{A}^n$. In other words, under the given conditions, ϕ is a bijective map that preserves the structure of $\mathbb{A}^n$.

3.3 Key Conditions

The main conditions for establishing the automorphism property are:

1. The Jacobian determinant $\det(J(\phi))$ is a non-zero constant polynomial.
2. The polynomials $f_1, f_2, \ldots, f_n$ defining ϕ are **well-defined** and **regular**.

These conditions ensure that the morphism ϕ exhibits both injective and surjective properties, making it an automorphism.

4 Proof of the Problem

We will provide a detailed proof covering injectivity, surjectivity, and the implications of the non-zero constant Jacobian determinant.

4.1 Injectivity

Suppose there exist two distinct points, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ in A^n such that $\phi(\mathbf{a}) = \phi(\mathbf{b})$. Consider the vector, $\mathbf{v} = \mathbf{b} - \mathbf{a}$.

For each component $i \in \{1, 2, \ldots.., n\}$, the equality $f_i(\mathbf{a} + \mathbf{v}) = f_i(\mathbf{a})$ implies:

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$$f_i(\mathbf{a}+\mathbf{v}) = f_i(\mathbf{a}) \tag{4.1}$$

$$f_i(a_1+v_1, a_2+v_2, \ldots, a_n+v_n) = f_i(a_1, a_2, \ldots, a_n)$$

Now, consider the *Taylor Expansion* of f_i around $\mathbf{a}$:

$$f_i(\mathbf{a}+\mathbf{v}) = f_i(\mathbf{a}) + \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\mathbf{a}) v_j + \textbf{higher-order terms}$$

Substituting this into the equation (4.1), we get:

$$f_i(\mathbf{a}) + \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\mathbf{a}) v_j + \textbf{higher-order terms} = f_i(\mathbf{a})$$

The higher-order terms vanish, and we are left with:

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\mathbf{a}) v_j = 0$$

This is equivalent to the dot product of the gradient vector $\nabla f_i(\mathbf{a})$ and $\mathbf{v}$:

$$\nabla f_i(\mathbf{a}) \cdot \mathbf{v} = 0$$

The non-zero constant Jacobian determinant implies that the gradients $\nabla f_i(\mathbf{a})$ are linearly independent. Therefore, $\mathbf{v} = \mathbf{0}$, and consequently, $\mathbf{a} = \mathbf{b}$.

Therefore, ϕ is **injective**.

4.2 Surjectivity

To establish surjectivity, we appeal to the *Implicit Function Theorem* (IFT). The non-zero constant Jacobian determinant ensures that the Jacobian matrix $J(\phi)$ is non-singular everywhere in $\mathbb{A}^n$. According to the IFT, this non-singularity implies local invertibility of ϕ at every point.

Since $\mathbb{A}^n$ is connected, local inverses can be combined to form a global inverse for ϕ. This is a consequence of the connectedness property, allowing the smooth patching together of local inverses.

Hence, ϕ is **surjective**.

Having established both injectivity and surjectivity, and understanding the implications of the non-zero constant Jacobian determinant, we conclude that ϕ is a bijective morphism and, therefore, an **Automorphism** on $\mathbb{A}^n$.

Remark 4.2.1. Our analysis focused on the significance of the Jacobian matrix and determinant, particularly when the determinant is a non-zero constant polynomial.

5 Conclusion

5.1 Observations

The key findings of our investigation can be summarized as follows:

5.1.1 Injectivity and Surjectivity

We established that a non-zero constant Jacobian determinant plays a crucial role in ensuring both injectivity and surjectivity of the morphism ϕ. The injectivity proof relied on the linearity of the Jacobian matrix and its connection to the vector $\mathbf{v} = \mathbf{b} - \mathbf{a}$. Meanwhile, the surjectivity proof invoked the Implicit Function Theorem, leveraging the non-singularity of the Jacobian matrix.

5.1.2 Global Invertibility

The connectedness of the affine space $\mathbb{A}^n$ allowed us to combine local inverses obtained from the Implicit Function Theorem to construct a global inverse for ϕ. This global invertibility is a fundamental characteristic of automorphisms.

5.1.3 Geometric Interpretation

The Jacobian matrix and determinant provided a geometric interpretation of the local behavior of ϕ. The Jacobian matrix represented the best linear approximation of ϕ near a given point, while the determinant measured the distortion of local volumes.

5.1.4 Non-Zero Constant Jacobian Determinant

The condition of a non-zero constant Jacobian determinant was pivotal in our proof. This condition ensured a consistent measure of invertibility across all points in $\mathbb{A}^n$, leading to a robust automorphism property for ϕ.

5.2 Implications and Future Directions

Our research contributes to the understanding of automorphisms in affine spaces and highlights the interplay between algebraic structures and geometric properties. The implications of our findings extend to various branches of mathematics, including algebraic geometry and differential geometry.

Future research could explore the generalization of these results to more abstract spaces and investigate the behavior of morphisms defined by different types of functions beyond polynomials. Additionally, applications in computer science, cryptography, and optimization could benefit from a deeper understanding of automorphisms in diverse mathematical contexts.

5.3 Final Remarks

In conclusion, our exploration of the automorphism property of ϕ sheds light on the intricate relationship between algebraic properties and geometric structures. The non-zero constant Jacobian determinant emerges as a key criterion for the robustness of automorphisms, paving the way for further inquiries into the rich tapestry of mathematical structures.

6 Future Directions

While our current research has provided valuable insights into the automorphism properties of the morphism $\phi : \mathbb{A}^n \to \mathbb{A}^n$, there are several promising directions for future exploration and study. In this section, we outline potential avenues for further research.

6.1 Generalization to Abstract Spaces

One natural extension is to investigate the generalization of our results to more abstract spaces beyond the affine space $\mathbb{A}^n$. Exploring automorphisms in spaces with different topological and algebraic structures could lead to a deeper understanding of the interplay between morphisms and the underlying mathematical space.

6.2 Morphisms Defined by Non-Polynomial Functions

Our current analysis focused on morphisms defined by polynomials. Future research could explore the behavior of morphisms defined by non-polynomial functions, including transcendental functions or piecewise-defined functions. Understanding the conditions under which such morphisms exhibit automorphism properties would expand the scope of our findings.

6.3 Applications in Computer Science and Cryptography

Investigating the practical applications of automorphisms, particularly in computer science and cryptography, presents an exciting avenue for research. Analyzing how automorphisms can be utilized in data encryption, optimization algorithms, or error-correcting codes could have significant implications for real-world problems.

6.4 Connections to Algebraic Geometry

Exploring the connections between automorphisms and algebraic geometry opens up a rich field of study. Investigating how the algebraic properties of morphisms relate to geometric properties, such as projective varieties or algebraic curves, could provide a more comprehensive understanding of the broader mathematical landscape.

6.5 Differential Aspects of Morphisms

Extending our analysis to include differential aspects of morphisms could offer valuable insights. Considering morphisms with differential structures, such as those involving differential equations or parametric representations, may reveal new phenomena and contribute to the development of differential geometry.

6.6 Quantum Automorphisms

A fascinating direction for future research involves exploring automorphisms in the context of quantum spaces. Investigating how quantum structures interact with morphisms and understanding the quantum analogs of automorphisms could contribute to the burgeoning field of quantum mathematics.

Statements and Declarations

Conflicts of Interest Statement

I as the sole author of this article certify that I have no affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affi liations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Data Availability Statement

I as the sole author of this aarticle confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

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