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Mohand Transform for Solution of Integral Equations and Abel's Equation

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Abstract: ***Integral transforms play a crucial role in determining the precise solution to differential equations and are distinguished by their simplicity and convenience. Several academics, beginning with Laplace, have formulated comprehensive equations for integral transforms. These transformations also hold significant importance in discovering precise answers to physical, technical, medicinal, and nuclear challenges, as well as in the fields of astronomy and economics. There are various types of integral transforms like Elzaki, Kamal, Aboodh, Mahgoub, sawi, Rishi, Anuj, Tarig, Kushare, Upadhyaya etc. were discussed about the solution of integral equations of linear volterra integral equations first kind and second kind, convolution type of volterra integral equations also Abel's integral equations. Here we discussed how the new integral transform Mohand is applicable to solve the integral equations and also convolution type of integral equations for unique solution of first and second kind of volterra integral equations, convolution type integral equations and also discussed Abel's integral equations.***

Keywords: Mohand transform, inverse Mohand transform, linear volterra integral equations (VIE), convolution model of VIE, Abel's equation, Laplace transform.

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1. Introduction

We use volterra integral equations in numerous fields, including mechanics, linear viscoelasticity, particle size statistics, heat transfer issue geometry, population dynamics, and epidemic research, Aggarwal and etc. all [1], [2], [3], [4], [5], [6], [7], [8], [9], [10] were solved the first and second kind volterra equations using various integral transformations (Rishi, Tarig, See, Anuj, Upadhyaya, Kamal, Elzaki and Aboodh). Recently Patil DP, Malpani SK, and Shinde PN [11], Sudhanshu Aggarwal, Swarg Deep Sharma, Aakansha Vyas [12], have examined the solution of volterra and convolution type integral equations and S. Aggarwal etc. all [13], [14], [15] have studied the solution Ables's integral equation. S M. Mohand and A. Mahgoub were introduced Mohand Transform, in 2017, here we discussed the solution for the second-kind of volterra integral equations, Abel's integral equations has been found, and which very easy and takes less time compare to other existed results.

2. Methodology

Let E be any set which is given as

$$E=\left\{f(t)/\exists K,K,k_1,k_2,|f(t)| \leq Ke^{-t/k_j}, where\ j\epsilon(-1)^j\times[0,\infty)\right\}$$

Then the Mohand transform of f (t) is given by $M\{f(t)\}=R(s)=s^2\int_0^\infty e^{-st}f(t)dt, s\epsilon(k_1,k_2)$

We have the following

(i) M{1}=s (ii) $M\{t^n\}=\frac{n!}{s^{n-1}}$ (iii) $M(e^{at})=\frac{s^2}{s-a}$ (iv) $M\{sinat\}=\frac{as^2}{s^2+a^2}$ (v) $M\{cosat\}=\frac{s^3}{s^2+a^2}$ (vi) $M\{f^1(t)\}=sR(s)-s^2f(0)$

Definition 2.1 The second kind of VIE of convolution type is given by
$\chi(x)=\varphi(x)+\lambda\int_a^x K(x-t)\,\chi(t)dt$, where λ is real or complex number.
where λ is non-zero real (or) complex number.

Here we have to determine $\chi(.)$

Definition 2.2 The form of Abel's integral equation is $f(t)=\int_0^t \frac{\varphi(x)}{(t-x)^\alpha}dt, 0<x<t.$

Theorem 2.3 Let $\varphi(s)$ and $\Psi(s)$ are Mohand transform of $f(t)$ and $g(t)$ respectively then

$$M\{f(t)*g(t)\}=\frac{1}{s^2}M\{f(t)\}.M\{g(t)\}$$

Where $*$ denote the convolution product.

Proof. We have $\omega(t)=\int_0^t f(\vartheta)g(t-\vartheta)d\vartheta$

Now consider $M(\omega(t))=s^2\int_0^\infty e^{-st}\left\{\int_0^t f(\vartheta)g(t-\vartheta)\,d\vartheta\right\}dt$

$$s^2\int_0^\infty\int_0^t e^{-st}f(\vartheta)g(t-\vartheta)d\vartheta dt$$

By change of order of integration, we obtain

$$M(\omega(t)) = s^2 \int_0^\infty \int_\vartheta^\infty e^{-st} f(\vartheta) g(t-\vartheta) d\vartheta dt$$

$$s^2 \int_\vartheta^\infty e^{-s\vartheta} f(\vartheta) \left\{ \int_0^\infty e^{-s(t-\vartheta)} g(t-\vartheta) dt \right\} d\vartheta$$

On putting $\delta = t - \vartheta$, then
$\int_0^\infty e^{-s\vartheta} f(\vartheta)\{s^2 \int_0^\infty e^{-s\delta} g(\delta) d\delta\} d\vartheta$

$$M(\omega(t)) = \int_0^\infty e^{-s\vartheta} f(\vartheta)\, d\vartheta\, \Psi(s) = \frac{1}{s^2}\varphi(s)\Psi(s).$$

This is the desired proof.

Corollary 2.4 Let $R(s)$ and $F(s)$ are the Mohand and Laplace transform of $f(t)$ respectively, then $F(s) = \frac{1}{s^2} R(s)$

Theorem 2.5 (Uniqueness)

Let F and G are the Mohand transform of f and g respectively. Then $g(t) = f(t)$

whenever $G(s) = F(s)$

Proof. Let us consider $G(s) = F(s)$

Then $M\{g(t), s\} = M\{f(t), s\} \Rightarrow \frac{1}{s^2} L\{g(t), s\} = \frac{1}{s^2} L\{f(t), s\}$

$$\Rightarrow L\{g(t), s\} = L\{f(t), s\}$$

Uniqueness of Laplace transforms. We have $g(t) = f(t)$.

3. Result and Discussion

Consider first kind of convolution type VIE, $f(y) = \int_0^y K(y-t)\, h(t) dt$

By using Theorem 2.3 and Mohand transform, we get
$F(s) = \frac{1}{s^2} M\{k(t)\}. M\{h(t)\}$
$F(s) = \frac{1}{s^2} K(s). H(s)$ then $H(s) = s^2 \frac{F(s)}{K(s)}$

By taking inverse Mohand transform we have $h(y) = M^{-1}\left\{s^2 \frac{F(s)}{K(s)}\right\}$

In the same manner let us take 2nd kind VIE, $h(y) = f(y) + \lambda \int_0^y K(y-t)\, h(t) dt$

By taking Mohand transform on both sides we have $H(s) = s^2 \frac{F(s)}{s^2 - \lambda K(s)}$

From above we get $h(y)$.

Now we can find the solution other problems of the above type with resolvent kernel

$$h(y) = f(y) + \lambda \int_0^y \Gamma(y-t)\, h(t) dt$$

This can be written as $H(s) = F(s) + \frac{\lambda s^2 K(s)}{s^2 - \lambda K(s)} F(s)$

It means that Γ(t) is the inverse Mohand transform of $\frac{\lambda s^2 K(s)}{s^2 - \lambda K(s)}$

With help of Mohand transform we can find the solution of Abel's integral equation

$$f(t) = \int_0^x \frac{\phi(x)}{(t-x)^\alpha} \text{ Where } 0 < \alpha < 1$$

The above equation can be written as $f(t) = g(t) * t^{-\alpha}$

Taking Mohand transform on both sides, we will have
$F(s) = \frac{1}{s^2} G(s) \frac{\Gamma(1-\alpha)}{s^{-\alpha-1}}$

$$\Rightarrow \frac{1}{s} G(s) = \frac{F(s) s^{-\alpha}}{\Gamma(1-\alpha)} \Rightarrow \frac{1}{s} G(s) = \frac{F(s)}{s^\alpha \Gamma(1-\alpha)} \Rightarrow \frac{1}{s} G(s) = \frac{F(s)\Gamma(\alpha)}{s^\alpha \Gamma(1-\alpha)\Gamma(\alpha)}$$

$$\Rightarrow \frac{1}{s} G(s) = \frac{F(s) \sin(\alpha\pi) \Gamma(\alpha)}{s^\alpha \pi \Gamma(\alpha)} \Rightarrow \frac{1}{s} G(s) = \frac{\sin(\alpha\pi)}{\pi} \frac{\Gamma(\alpha)}{s^\alpha} F(s)$$

This can be written as in terms of convolution, $\frac{1}{s} G(s) = \frac{\sin(\alpha\pi)}{\pi} M\{t^{\alpha-1} * f(t)\}$

$$\frac{1}{s} G(s) = \frac{\sin(\alpha\pi)}{\pi} M\left\{ \int_0^t (t-x)^{\alpha-1} f(x)\, dx \right\}$$

Let $\Psi(t) = \int_0^t (t-x)^{\alpha-1} f(x)\, dx$ and we have $\Psi(0) = 0$.

But we have $M\{\Psi^1(t)\} = sM\{\Psi(t)\} - s^2\Psi(0) \Rightarrow M\{\Psi^1(t)\} = sM\{\Psi(t)\}$

From above we get $g(t) = \frac{\sin(\alpha\pi)}{\pi} \frac{d}{dt}\left\{\int_0^t (t-x)^\alpha f(x)\, dx\right\}$

Where g(t) is the solution of Abel's integral equation.

4. Applications to Integral Equations

We demonstrated some examples on convolution type first, second kind of VIE and also discussed resolvent kernels by using Mohand Transform.

Example 4.1 Solve $sint = \int_0^t e^{t-x} \varphi(x) dx$

Solution. It is the homogeneous voltera integral equation of second kind.

Taking Mohand transform on both sides we get

$$M\{sint\} = M\left\{ \int_0^t e^{t-x} \varphi(x) dx \right\}$$

From convolution theorem, $M\{sint\} = \frac{1}{s^2}M\{e^t\}M\{\varphi(t)\}$

$$\Rightarrow \frac{s^2}{s^2+1} = \frac{1}{s^2}\frac{s^2}{s-1}\varphi(s) \Rightarrow \varphi(s) = \frac{s^3-s^2}{s^2+1}$$

By taking inverse Mohand transform, $\varphi(t) = cos(t) - sin(t)$

Example 4.2 Solve $\varphi^1(t) = \sin(t) + \int_0^t \varphi(t-x)cosxdx$, where $\varphi(0) = 0$

Solution. Taking Mohand transform and using convolution theorem we have

$$M\{\varphi^1(t)\}=M\{sint\} + \frac{1}{s^2}M\{\varphi(t)\}M\{cost\}$$

From Mohand transform of derivatives

$$sM\{\varphi(t)\} - s^2\varphi(0) = \frac{s^2}{s^2+1} + \frac{1}{s^2}M\{\varphi(t)\}\frac{s^3}{s^2+1}$$

$$\Rightarrow \left[s - \frac{s}{s^2+1}\right]M\{\varphi(t)\} = \frac{s^2}{s^2+1} \Rightarrow M\{\varphi(t)\} = \frac{1}{s}$$

Then from inverse mohand transform $\varphi(t) = \frac{t^2}{2}$

Example 4.3 Solve $\varphi(x) = \sin(x) + 2\int_0^t e^{t-x}\varphi(t)dt$

Solution. Take Mohand transform on both sides, we have

$$M\{\varphi(x)\} = M\{sinx\} + \frac{2}{s^2}M\{e^t\}M\{\varphi(x)\}$$

$$\Rightarrow M\{\varphi(x)\} = \frac{s^2}{s^2+1} + \frac{1}{s^2}\frac{s^2}{s-1}M\{\varphi(x)\}$$

$$\Rightarrow \left(\frac{s-3}{s-1}\right)M\{\varphi(x)\} = \frac{s^2}{s^2+1} \Rightarrow M\{\varphi(x)\} = \frac{s^2}{s^2+1}\left(\frac{s-1}{s-3}\right)$$

$$\Rightarrow M\{\varphi(x)\} = \frac{1}{5}\frac{s^2}{s-3} + \frac{1}{5}\left(2\frac{s^2}{s^2+1} - \frac{s^3}{s^2+1}\right)$$

By taking inverse transform $\varphi(x) = \frac{1}{5}e^{3x} + \frac{2}{5}sinx - \frac{1}{5}cosx$

Example 4.4 Find the resolvent kernel of $\varphi(t) = \Psi(t) + \int_0^t e^{t-\vartheta}\varphi(\vartheta)d\vartheta$

Solution. Let the required resolvent kernel is $R(t-x)$, so consider

$$M\{R(t)\} = \frac{s^2K(s)}{s^2 - K(s)}, where\ \lambda = 1$$

$$M\{R(t)\} = \frac{s^2\frac{s^2}{(s-1)}}{s^2 - \frac{s^2}{(s-1)}} = \frac{s^2}{s-2}$$

By taking inverse Mohand transform we get $R(t) = e^{2t}$.

Then the resolvent kernel is $R(t-x) = e^{2(t-x)}$

Example 4.5 Solve $\sqrt{t} = \int_0^t \frac{\varphi(x)}{\sqrt{(t-x)}}dx$

Solution. The above equation can be written as $\sqrt{t} = \varphi(t) * t^{\frac{-1}{2}}$

Taking Mohand transform on both sides

$$M\left\{t^{1/2}\right\} = \frac{1}{s^2}M\{\varphi(t)\}M\left\{t^{-1/2}\right\}$$

$$\Rightarrow \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{1/2-1}} = \frac{1}{s^2}M\{\varphi(t)\}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-1/2-1}} \Rightarrow M\{\varphi(t)\} = \frac{s}{2}$$

By taking inverse transform, we get $\varphi(t) = \frac{1}{2}$

Example 4.6 Solve $\int_0^t \frac{\varphi(x)}{(t-x)^{1/3}} = t(1+t)$

Solution. This can be written as $\varphi(t) * t^{-1/3} = t + t^2$

Taking Mohand transform on both sides

$$M\left\{\varphi(t) * t^{-1/3}\right\} = M(t + t^2)$$

$$\Rightarrow M\left\{\varphi(t) * t^{-1/3}\right\} = M(t) + M(t^2)$$

$$\Rightarrow \frac{1}{s^2}M\{\varphi(t)\}\frac{\Gamma(-1/3+1)}{s^{-1/3-1}} = 1 + \frac{2}{s} \Rightarrow \frac{\Gamma(2/3)}{s^{2/3}}M\{\varphi(t)\} = 1 + \frac{2}{s}$$

$$\Rightarrow M\{\varphi(t)\} = \frac{1}{\Gamma(\frac{2}{3})}s^{2/3}\left(1 + \frac{2}{s}\right)$$

By taking inverse Mohand transform we get $\varphi(t) = \frac{3\sqrt{3}}{\pi}t^{\frac{1}{3}}(1 + \frac{3}{2}t)$

5. Conclusion

We have successfully used the Mohand Transform in this manuscript to solve the Volterra equation of the second kind. Here, we obtain a convolution type kernel for the VIE of the second kind. We suggested using the second form of volterra equations in a few different scientific and engineering fields, and we determined their precise solution. This method can be used to solve other kinds of linear integral equations and requires less calculation work in a very short amount of time.

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