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Degree Of Approximation of Function in the Holder Metric By $(N\ p_n)(N\ p_n)$ Means

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Abstract: ***In this paper we establish a theorem on degree of approximation of function f in Hölder metric by $(N\ p_n)(N\ p_n)$ means.***

Keywords: Fourier Series, Hölder metric, $(N\ p_n)$ means, degree of approximation, summability method

MSC: 41A24,41A25,42B05,42B08

1. Introduction

In recent decades researchers have increasingly devoted to examining error estimation within various function spaces by employing various single and product summability means. Approximation theory has gained popularity due to its transformative and contributive impact on signal approximation. In 1928 Alexits [1] obtained degree of approximation of function in Holder metric by Cesaro means of its Fourier series. Later on, several researcher like Prosdorff [5], Mohapatra and Chandra [4], Chandra [2], Das et al [3] obtained degree of approximation of function in Holder metric of its Fourier series in Fejer mean, Matrix mean, Euler means and Borel means respectively. Very recently Sinha and Shrivastav [9], Rathore and Shrivastav [8] and Rathore [9] obtained degree of approximation of function in Holder metric of its Fourier series by employing different product means $(N\ p_n)$ (E q) means, (C 1)F(a q) means, (C 1) (e c) means respectively. In this context ,we obtained result concerning the degree of approximation of function by using Holder metric by employing $(N\ p_n)$ $(N\ p_n)$ product summability means.

2. Preliminaries and notations

Definition 2.1: Holder metric spaces:

Let $C_{2\pi}$ denote the Banach space of all 2π periodic continuous function defined on $[0\ 2\pi]$ under the supremum norm. For $0< \alpha \le 1$, and some positive constant M then Holder Metric space H_α is given by the following:

$$H_\alpha=\{f \in C_{2\pi}: |f(x)-f(y)| \le M|x-y|^\alpha)\} \qquad (1)$$

The space H_α is a Banach space with the norm $\|\,.\,\|_\alpha$ defined by

$$\| f\|_\alpha = \| f\|_c + + \sup_{x,y}[\Delta^\alpha f(x\ y)] \qquad (2)$$

where $\| f\|_c = \sup_{-\pi\le x\le\pi} |f(x)|$

$$\text{and } \Delta^\alpha f(x\ y) = \frac{|f(x)-f(y)|}{|x-y|^\alpha} (x \ne y) \qquad (3)$$

we shall use the convention that $\Delta^0 f(x\ y)$=0

The metric induced by norm in (2) on H_α is called the Holder Metric.

Definition 2.2: Fourier series:

Let f be a 2π periodic function integrable in the sense of Lebesgue then the Fourier series of f at any point x is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad (4)$$

where a_0 ,a_n ,b_n are Fourier coefficients

The n^{th} partial sum of Fourier series is represented by,

$$s_n(x) - f(x) = \frac{1}{2\pi}\int_0^\pi \emptyset_x(t) \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt \qquad (5)$$

$$\text{where } \emptyset_x(t) = f(x+t) + f(x-t) - 2f(x) \qquad (6)$$

Definition 2.3: Summability means: Let $v_1 + v_2 + v_3 \ldots\ldots\ldots\ldots\ldots.. = \sum_{n=0}^{\infty} v_n$ be an infinite series with the sequence of its partial sum $\{s_n\}$.

$$s_n=\sum_{k=0}^{n} v_k$$

If $s_n \to s$ as n→ ∞ then the infinite series$\sum_{n=0}^{\infty} v_n$ or the sequence $\{s_n\}$ is said to summable by to a definite number s.

Definition 2.4: Norlund Mean or $(N\ ,p_n)$ Mean:

Let $\{p_n\}$ be sequence of real or constant such that

$$P_n = p_0 + p_1 + p_2 + \ldots\ldots\ldots\ldots p_n$$
$$P_n = \sum_{i=0}^{n} p_i \ne 0 \text{ and } p_{-1}=0$$

Norlund $(N\ ,p_n)$ means of sequence of partial sum $\{s_n\}$ is described by,

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\, s_k \qquad (7)$$

If $t_n \to s$ as n→ ∞ then the infinite series$\sum_{n=0}^{\infty} v_n$ or the sequence of partial sum $\{s_n\}$ is said to summable by $(N\ ,p_n)$ method to a definite number s.

Definition 2.5: Norlund -Norlund Product Means:
If (N, p_n) method is superimposed on (N, p_n) method a new method $(N, p_n)(N, p_n)$ is obtained. It is known as Norlund - Norlund product summability method. The Norlund – Norlund product means t_n^{NN} of partial sum of sequence $\{s_n\}$ is given by,

$$t_n^{NN} = \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m}\left\{\frac{1}{P_m}\sum_{k=0}^{m} p_{m-k}\ s_k\right\} \quad (8)$$

If $t_n^{NN} \to s$ as $n \to \infty$ then the infinite series $\sum_{n=0}^{\infty} v_n$ or the sequence of partial $\{s_n\}$ is said to summable by $(N, p_n)\ (N, p_n)$ method to a definite number s.

Definition 2.6: Degree of Convergence: The degree of convergence of a summation method to a given function f indeed measure how quickly the γ_n approaches to the function f, which is defined as,

$$\|f - \gamma_n\| = O\left(\frac{1}{\lambda_n}\right) \quad (9)$$

where γ_n is trigonometric polynomial of degree n and $\lambda_n \to \infty$ as $n \to \infty$.

Throughout this paper we use the following notation.

$$\varphi_x(t) = f(x+t) + f(x-t) - 2f(x) \quad (10)$$

$$Z_n(t) = \frac{1}{2\pi P_n}\sum_{m=0}^{n} p_{n-m}\left\{\frac{1}{P_m}\sum_{k=0}^{m} p_{m-k}\ \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\} \quad (11)$$

3. Known Results

In 1928 Alexits [1] proved the following result.

Theorem 3.1: If $f \in C_{2\pi} \cap Lip\alpha$ $(0 < \alpha \le 1)$, then

$$\left\|\sigma_n^{\delta}(f) - f\right\| = \{n^{-\alpha}\ logn\}, \quad (12)$$

where $0 < \alpha \le \delta \le 1$ and $\sigma_n^{\delta}(f;x)$ is (C, δ) means of $(s_n(f;x))$.

In 2019 Sinha and Shrivastav [10] proved following results.

Theorem 3.2: For $0 < \beta \le \alpha \le 1$ and $f \in H_\alpha$ then for $n > 1$

$$\|t_n(f) - f\|_\beta == O\left((n)^{\beta-\alpha}(logn)^{\frac{\beta}{\alpha}}\right) \quad (13)$$

In 2023 Rathore and Shrivastav [9] proved following result.
Theorem 3.3: : For $0 < \beta \le \alpha \le 1$ and $f \in H_\alpha$ then

$$\|C_n^1 e_n^c - f(x)\|_\beta = O\left((n+1)^{\beta-\alpha}(\log(n+1))^{\frac{\beta}{\alpha}}\right) \quad (14)$$

where $C_n^1 e_n^c$ is the product summability (C 1) (e c) means of $(s_n(f;x))$.

4. Main Result

Theorem 4.1 For $0 < \beta \le \alpha \le 1$ and $f \in H_\alpha$ then degree of approximation of f by $(N\, p_n)(N\, p_n)$ means of Fourier series is given by,

$$\|t_n^{NN}(f) - f\|_\beta = O\left((n+1)^{\beta-\alpha}(logn)^{\frac{\beta}{\alpha}}\right) \quad (15)$$

where t_n^{NN} is $(N\, p_n)(N\, p_n)$ product summability means of $(s_n(f;x))$.

5. Required Lemma

To establish main theorem following lemmas are required.

Lemma 5.1:If $\varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$ then for $f \in H_\alpha$ and $0 < \alpha \le 1$ we have

$$\left|\varphi_x(t) - \varphi_y(t)\right| = M|x-y|^\alpha \quad (16)$$

$$\left|\varphi_x(t) - \varphi_y(t)\right| = M(|t|^\alpha) \quad (17)$$

Lemma 5.2: For $0 \le t \le \frac{\pi}{n}$ and sinnt≤nsint then

$$Z_n(t) = O(n+1) \quad (18)$$

Proof:

$$|Z_n(t)| = \left|\frac{1}{2\pi P_n}\sum_{m=0}^{n} p_{n-m}\left\{\frac{1}{P_m}\sum_{k=0}^{m} p_{m-k}\ \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi P_n}\sum_{m=0}^{n} p_{n-m}\left\{\frac{1}{P_m}\sum_{k=0}^{m} p_{m-k}\ (2k+1)\frac{\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}$$

$$\le \frac{1}{2\pi P_n}\sum_{m=0}^{n} p_{n-m}\ (2m+1)\left\{\frac{1}{P_m}\sum_{k=0}^{m} p_{m-k}\right\}$$

$$\le \frac{(2n+1)}{2\pi P_n}\sum_{m=0}^{n} p_{n-m}$$

$$= O(n+1)$$

Lemma 5.3: For $\frac{\pi}{n} \le t \le \pi$, $\sin\frac{t}{2} \ge \frac{t}{\pi}$ and sint≤ 1 then

$$Z_n(t) = O\left(\frac{1}{t}\right) \quad (19)$$

Proof:

$$|Z_n(t)| = \left|\frac{1}{2\pi P_n}\sum_{m=0}^{n} p_{n-m}\left\{\frac{1}{P_m}\sum_{k=0}^{m} p_{m-k}\ \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi P_n}\sum_{m=0}^{n} p_{n-m}\left\{\frac{1}{P_m}\sum_{k=0}^{m} p_{m-k}\ \frac{1}{\frac{t}{\pi}}\right\}$$

$$\le \frac{1}{2tP_n}\sum_{m=0}^{n} p_{n-m}\left\{\frac{1}{P_m}\sum_{k=0}^{m} p_{m-k}\right\}$$

$$\le \frac{1}{2tP_n}\sum_{m=0}^{n} p_{n-m}$$

$$= O\left(\frac{1}{t}\right)$$

6. Proof of the theorem

Let $s_k(x)$ denote the k^{th} partial sum of Fourier series ( ) then we have,

$$s_k(x) - f(x) = \frac{1}{2\pi}\int_0^\pi \varphi(x,t)\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt \quad (20)$$

(N, p_n) transform of $s_k(x)$ is denoted by t_n^N and given by

$$t_n^N(x) - f(x) = \frac{1}{2\pi P_n}\int_0^\pi \varphi(x,t)\left\{\sum_{k=0}^{n} p_{n-k}\ \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\} \quad (21)$$

The $(N, p_n)\ (N, p_n)$ transform of $s_k(x)$ is given by,
$t_n^{NN}(x) - f(x) =$

$$\frac{1}{2\pi P_n}\sum_{m=0}^{n}\left[P_{n-m}\int_0^\pi \varphi(x,t)\frac{1}{P_m}\left\{\sum_{k=0}^{m} p_{m-k}\ \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt\right] \quad (22)$$

$$= \int_0^\pi \varphi(x,t)\ Z_n(t)\,dt$$

$$= \int_0^{\frac{\pi}{n}} \varphi(x,t)\ Z_n(t)dt + \int_{\frac{\pi}{n}}^{\pi} \varphi(x,t)\ Z_n(t)dt \quad (23)$$

Now $T_n(x) = |\ t_n^{NN}(x) - f(x)|$ (24)
and $T_n(x, y) = |T_n(x) - T_n(y)|$ (25)

$|T_n(x) - T_n(y)| = \int_0^{\frac{\pi}{n}} |\varphi_x(t) - \varphi_y(t)|\, |Z_n(t)|dt$

$+ \int_{\frac{\pi}{n}}^{\pi} |\varphi_x(t) - \varphi_y(t)|\, |Z_n(t)|dt$ (26)

$= J_1 + J_2$ (27)

Using lemma (5.1)and lemma(5.3) we get,

$J_1 = \int_0^{\frac{\pi}{n}} |\varphi_x(t) - \varphi_y(t)|\, |Z_n(t)|dt$

$= O(n+1) \int_0^{\frac{\pi}{n}} t^\alpha\, dt$

$= O(n+1)(t^{\alpha+1})_0^{\frac{\pi}{n}}$

$= O(n+1)^{-\alpha}$ (28)

Now $J_2 = \int_{\frac{\pi}{n}}^{\pi} |\varphi_x(t) - \varphi_y(t)|\, |Z_n(t)|dt$

$= \int_{\frac{\pi}{n}}^{\pi} t^\alpha \frac{1}{t}\, dt$

$= (t^\alpha)_{\frac{\pi}{n}}^{\pi}$

$= O(n)^{-\alpha}$ (29)

Again by using lemma5.1 and lemma 5.2 we have,

$J_1 = \int_0^{\frac{\pi}{n}} |\varphi_x(t) - \varphi_y(t)|\, |Z_n(t)|dt$

$= O(|x-y|^\alpha\, (n+1))$ (30)

Again by using lemma 5.1 and lemma 5.3

$J_2 = \int_{\frac{\pi}{n}}^{\pi} |\varphi_x(t) - \varphi_y(t)|\, |Z_n(t)|dt$

$= O\left(|x-y|^\alpha \int_{\frac{\pi}{n}}^{\pi} \frac{1}{t}\, dt\right)$

$= O(|x-y|^\alpha \log n)$ (31)

Now for k=1, 2,3 …..and for $0 \le \beta < \alpha \le 1$ $J_k^{1-\frac{\beta}{\alpha}}$ and $J_k^{\frac{\beta}{\alpha}}$ we observe from (28) and (30) and get ,

$J_1 = O\left[((n+1)^{-\alpha})^{1-\frac{\beta}{\alpha}} (|x-y|^\alpha\, (n+1))^{\frac{\beta}{\alpha}}\right]$

$= O\left[(n+1)^{\beta-\alpha+\frac{\beta}{\alpha}}\, |x-y|^\beta\right]$ (32)

From (29) and (31) we get,

$J_2 = O\left[((n)^{-\alpha})^{1-\frac{\beta}{\alpha}} (|x-y|^\alpha \log n)^{\frac{\beta}{\alpha}}\right]$

$= O\left[(n)^{\beta-\alpha}\, |x-y|^\beta\, (\log n)^{\frac{\beta}{\alpha}}\right]$ (33)

From (32) and (33) we get

$|T_n(x) - T_n(y)| = O\left[(n+1)^{\beta-\alpha+\frac{\beta}{\alpha}}\, |x-y|^\beta\right] + O\left[(n)^{\beta-\alpha}\, |x-y|^\beta\, (\log n)^{\frac{\beta}{\alpha}}\right]$

$= O\left[(n+1)^{\beta-\alpha}\, |x-y|^\beta\, (\log n)^{\frac{\beta}{\alpha}}\right]$ (34)

And

$\Delta^\beta[T_n(x, y)] = \frac{|T_n(x)-T_n(y)|}{|x-y|^\beta}, \quad (x \neq y)$ (35)

$= O\left[(n+1)^{\beta-\alpha}(\log n)^{\frac{\beta}{\alpha}}\right]$ (36)

$\sup_{x \neq y} \Delta^\beta[T_n(x, y)] = O\left[(n+1)^{\beta-\alpha}(\log n)^{\frac{\beta}{\alpha}}\right]$ (37)

Now $\|T_n\| = O(n+1)^{-\alpha}$ (38)

Combining (37) and (38) we get

$\|t_n^{NN} - f\|_\beta = O\left[(n+1)^{\beta-\alpha}(\log n)^{\frac{\beta}{\alpha}}\right]$

This completes the proof of the theorem.

7. Corollaries

Corollary7.1 If $(N\, p_n)(N\, p_n)$ means reduces to$(N\, p_n)$(E q) means, then degree of approximation of f ∈ H_α for $0 < \beta \le \alpha \le 1$ by $(N\, p_n)$ (E q) means of Fourier series is given by,

$\|t_n^{NE}(f) - f\|_\beta = O\left((n+1)^{\beta-\alpha}(\log n)^{\frac{\beta}{\alpha}}\right)$

Corollary7.2: If $(N\, p_n)(N\, p_n)$ means reduces to (E q) $(N\, p_n)$ means, then degree of approximation of f ∈ H_α for $0 < \beta \le \alpha \le 1$ by (E q)$(N\, p_n)$ means of Fourier series is given by ,

$\|t_n^{EN}(f) - f\|_\beta = O\left((n+1)^{\beta-\alpha}(\log n)^{\frac{\beta}{\alpha}}\right)$

Corollary7.3: If $(N\, p_n)(N\, p_n)$ means reduces to (E q) (E q) means, then degree of approximation of f ∈ H_α for $0 < \beta \le \alpha \le 1$ by (E q) (E q) means of Fourier series is given by ,

$\|t_n^{EE}(f) - f\|_\beta = O\left((n+1)^{\beta-\alpha}(\log n)^{\frac{\beta}{\alpha}}\right)$

Corollary7.4: If $(N\, p_n)(N\, p_n)$ means reduces to (E q) (E 1) means ,then degree of approximation of f ∈ H_α for $0 < \beta \le \alpha \le 1$ by (E q) (E 1) means of Fourier series is given by ,

$\left\|t_n^{EE_1}(f) - f\right\|_\beta = O\left((n+1)^{\beta-\alpha}(\log n)^{\frac{\beta}{\alpha}}\right)$

Corollary7.5: If $(N\, p_n)(N\, p_n)$ means reduces to (E 1) (E q) means ,then degree of approximation of f ∈ H_α for $0 < \beta \le \alpha \le 1$ by (E 1) (E q) means of Fourier series is given by ,

$\left\|t_n^{E_1E}(f) - f\right\|_\beta = O\left((n+1)^{\beta-\alpha}(\log n)^{\frac{\beta}{\alpha}}\right)$

Corollary7.6: If $(N\, p_n)(N\, p_n)$ means reduces to (E 1) (E 1) means, then degree of approximation of f ∈ H_α for $0 < \beta \le \alpha \le 1$ by (E 1) (E 1) means of Fourier series is given by,

$\left\|t_n^{E_1E_1}(f) - f\right\|_\beta = O\left((n+1)^{\beta-\alpha}(\log n)^{\frac{\beta}{\alpha}}\right)$

8. Conclusion

In this paper we have determined degree of approximation of function in Holder metric space by employing $(N\, p_n)(N\, p_n)$ product summability of its Fourier series. The product summability means $(N\, p_n)$(E q), (E q) $(N\, p_n)$, (E q) (E q), (E q) (E 1),(E 1) (E q)and (E 1) (E 1) are particular cases of $(N\, p_n)(N\, p_n)$means, so our result also hold for these means which are presented in the form of corollaries. Thus our result generalizes several known results.

References

[1] Alexits, G., Convergence problem on Orthogonal series, Pergamon Press, New York, (1961).
[2] Chandra, P, On the generalized Fejer mean in the metric of Holder Metric, Mathematics nachrichten,109 (1) (1982),39-45
[3] Das, G. Ghosh, T. Ray,B.K.,Degree of approximation of function by their Fourier series in the generalized Holder metric ,Proceeding of the Indian Academy of Science ,106(2)(1996),139-153.
[4] Mohapatra, R. N., Chandra, P, Degree of approximation of function in the Holder Metric, Acta Math.Hung,41(1983),67-76.
[5] Prossdorff, S, Zur konergenz der Fourierreithen Holder Stetiger Funkktionmen, Mathematische Nachrichten,69 (1) (1975),7-14.

[6] Rathore, H.L., Shrivastav, U. K., Mishra, V. N., On approximation of continuous function in the Holder metric by (C 1) [F d_n] means of its Fourier series, Jnanabha,51(2) (2021),161-167.
[7] Rathore, H.L., Shrivastav, U.K., and Mishra, V. N., On approximation of continuous function in the Holder metric by (C 2) (E q) means of its Fourier series, Material Today Proceeding, 57(5) (2022),2026-2030.
[8] Rathore, H. L., Shrivastav, U.K. and Mishra, V. N., Degree of approximation of continuous function in the Holder metric by (C 1) F (a, q) means of its Fourier series, Ganita ,72(2)(2022),19-30.
[9] Rathore, H.L, Degree of approximation of function in the Holder metric (C1) (e c) means of its Fourier series. International Journal of Mathematics and Application, 11(4) (2023),99-109.
[10] Sinha, S., Shrivastav,U.K., Degree of approximation of function in the Holder metric by $(N\ p_n)(E\ q)$ means .International Journal of Mathematics Trend and Technology,Vol 65,Issued 4,2019,101-106.
[11] Titchmarsh, E.C., A theory of function, Oxford University Press, London,(1978),402-403.
[12] Zygmund, A, Trignometric Series Vol.1, Cambridhe University Press, New York, USA, (1959).

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