International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Quest of Some Advanced Results among Jacobsthal and Jacobsthal-Lucas Numbers

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Abstract: ***In this paper, the properties of Jacobsthal and Jacobsthal - Lucas sequences associated with an aspect of congruence and divisibility are analyzed using several relations that hook up these two sequences.***

Keywords: Jacobsthal sequences, Jacobsthal - Lucas sequences, Divisibility

1. Introduction

There is a wide literature about the Jacobsthal and Jacobsthal-Lucas sequences as well as others, including the Fibonacci sequence, Pell sequence, and Pell-Lucas sequence, where numerous features are researched. The Fibonacci Q-matrix and the Jacobsthal Lucas E- and R-matrices were defined by the authors in [2]. Moreover, the Jacobsthal-Lucas E-matrix and R-matrix are used to find certain equalities and a Binet-like formula for the Jacobsthal and Jacobsthal-Lucas numbers. Ahmet Daşdemir [3] investigated a novel complex sum formula and the identities between the Jacobsthal numbers and matrices. Authors [7] compared the precise and divisibility aspects of the Pell and Associated Pell numbers. Congruences and Divisibility properties of Pell and Pell-Lucas numbers deliberated by Pandichelvi. V and Sandhya. P in [8].For more information about Jacobsthal and Jacobsthal - Lucas sequences, one may refer $[3-6]$.

In this work, the characteristics of the Jacobsthal and Jacobsthal - Lucas sequences related to congruence and divisibility are examined using several relations that connect these two sequences.

2. Properties of Jacobsthal and Jacobsthal - Lucas numbers holding summation

The Jacobsthal sequence is characterized by $\{\mathcal{J}_n\}$ and labelled by the following recurrence relation $\mathcal{J}_n = \mathcal{J}_{n-1} + 2\mathcal{J}_{n-2}$ for $n \geq 2$ with initial condition $\mathcal{J}_0 = 0$ and $\mathcal{J}_1 = 1$. The Jacobsthal-Lucas sequence is regarded as $\{\mathcal{L}_m\}$ and categorized by the same recurrence relation $\mathcal{L}_m = \mathcal{L}_{m-1} + 2\mathcal{L}_{m-2}$ for $n \geq 2$ with initial condition $\mathcal{L}_0 = 2$ and $\mathcal{L}_1 = 1$.

The Binet's formula for the Jacobsthal and Jacobsthal – Lucas numbers are $\frac{1}{3}(2^n - (-1)^n)$ and $[2^n + (-1)^n]$ respectively.

Theorem 2.1
If R is a square matrix with $R^2 = R + 2I$, then $R^n = \mathcal{J}_n R + 2\mathcal{J}_{n-1} I$ for every integer n.

Proof
Let $Z[i] = \{pi + 2qI;\ p,\ q \in Z\}$ and $Z[R] = \{pR + 2qI; p, q \in Z\}$

Define a function $\varphi: Z[i] \to Z[R]$ by $\varphi(pi + qI) = pR + qI$. Then, φ is a ring isomorphism. Moreover, it is clear that $\varphi(i) = R$ and $\varphi(\mathcal{L}_m) = 2\mathcal{L}_m I$.
Therefore, $R^n = (\varphi(i))^n = \varphi(i^n) = \varphi(\mathcal{J}_n i + 2\mathcal{J}_{n-1}) = \mathcal{J}_n R + 2\mathcal{J}_{n-1} I$

Corollary 2.1.1
Let B = $\begin{pmatrix} \frac{1}{2} & \frac{9}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and then $B^n = \begin{pmatrix} \frac{\mathcal{L}_n}{2} & \frac{9\mathcal{L}_n}{2} \\ \frac{\mathcal{J}_n}{2} & \frac{\mathcal{L}_n}{2} \end{pmatrix}$ for every $n \in Z$

Proof
It is determined from Theorem 2.1 that $R^2 = R + 2I$.
and it is well recognized that $\varphi : Z[i] \to Z[R]$ by $\varphi(pi + qI) = pR + qI$ is a ring isomorphism.

Further, the identities from these conditions are discovered by

$$i^{2x} - \mathcal{L}_x i^x + (-2)^x = 0 \quad (1)$$

$$i^{2x} - 3\mathcal{J}_x i^x - (-2)^x = 0 \quad (2)$$

Application of the function φ in (1) and (2) respectively provides that

$$B^{2x} - \mathcal{L}_x B^x + (-2)^x = 0 \quad (3)$$

$$B^{2x} - T\mathcal{J}_x B^x - (-2)^x = 0 \quad (4)$$

where

$$T = \varphi(3) = \varphi(2B - 1) = 2B - I = \begin{pmatrix} 0 & 9 \\ 1 & 0 \end{pmatrix}$$

Theorem 2.3

If $y \in N$ and $x, u \in Z$, then

$$\mathcal{L}_{2xy+u} = (-1)^{(x+1)y} 2^{xy} \sum_{t=0}^{y} \binom{y}{t} (-1)^{(x+1)t}\, 2^{-xt}\, \mathcal{L}_x^t\, \mathcal{L}_{xt+u}$$

$$\mathcal{J}_{2xy+u} = (-1)^{(x+1)y}\, 2^{xy} \sum_{t=0}^{y} \binom{y}{t} (-1)^{(x+1)t}\, 2^{-xt}\, \mathcal{L}_x^t\, \mathcal{J}_{xt+u}$$

Proof
From (3) in Theorem 2.2, it is noted by

$$B^{2x} = \mathcal{L}_x B^x - (-2)^x I \qquad (5)$$

Levitation of y^{th} power and multiplying B^u on both sides of (5) yields that

$$B^{2xy+u} = (-1)^{(x+1)y} 2^{xy} \sum_{t=0}^{y} \binom{y}{t} (-1)^{(x+1)t} 2^{-xt} \mathcal{L}_x^t B^{xt+u}$$

Thus, it follows that

$$\mathcal{L}_{2xy+u} = (-1)^{(x+1)y} 2^{xy} \sum_{t=0}^{y} \binom{y}{t} (-1)^{(x+1)t} 2^{-xt} \mathcal{L}_x^t \mathcal{L}_{xt+u}$$

$$\mathcal{J}_{2xy+u} = (-1)^{(x+1)y} 2^{xy} \sum_{t=0}^{y} \binom{y}{t} (-1)^{(x+1)t} 2^{-xt} \mathcal{L}_x^t \mathcal{J}_{xt+u}$$

Corollary 2.3.1

For all $y \in N$ and $x, u \in Z$,

$$L_{2xy+u} \equiv (-1)^{(x+1)y} 2^{xy} \mathcal{L}_u (\mathrm{mod}\ \mathcal{L}_x) \qquad (6)$$

$$\mathcal{J}_{2xy+u} \equiv (-1)^{(x+1)y} 2^{xy} \mathcal{J}_u (\mathrm{mod}\ \mathcal{L}_x) \qquad (7)$$

Remarks:
1) Since $T = 2M - I = M + 2M^{-1}$, $M^x T = M T^x$ for all $x \in Z$
2) $T^2 = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} = 9I$ and $\begin{pmatrix} 0 & 9 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} 9k & 9l \\ i & j \end{pmatrix}$

Theorem 2.5
If $y \in N$ and $x, u \in Z$, then

$$\mathcal{L}_{2xy+u} = (-2)^{xy} \left\{ \sum_{t=o}^{\left\lfloor \frac{y}{2} \right\rfloor} \binom{y}{2t} 9^t \mathcal{J}_x^{\ 2t} \mathcal{L}_{2xt+u} + \sum_{t=o}^{\left\lfloor \frac{(y-1)}{2} \right\rfloor} \binom{y}{2t+1} (-1)^{xt} 9^t \mathcal{J}_x^{\ 2t+1} \mathcal{J}_{2xt+x+u} \right\}$$

$$\mathcal{J}_{2xy+u} = (-2)^{xy} \left\{ \sum_{t=o}^{\left\lfloor \frac{y}{2} \right\rfloor} \binom{y}{2t} 9^t \mathcal{J}_x^{\ 2t} \mathcal{J}_{2xt+u} + \sum_{t=o}^{\left\lfloor \frac{(y-1)}{2} \right\rfloor} \binom{y}{2t+1} (-1)^{xt} 9^t \mathcal{J}_x^{\ 2t+1} \mathcal{L}_{2xt+x+u} \right\}$$

Proof
From (4), it tracks that

$$B^{2x} = T\mathcal{J}_x B^x + (-2)^x I$$

Therefore,

$$B^{2xy+u} = (T\mathcal{J}_x B^x + (-2)^x I)^y B^u$$

$$= \left\{ \sum_{t=o}^{y} \binom{y}{t} [(-1)^x 2^x I]^{y-t} [T\mathcal{J}_x B^x]^t \right\} B^u$$

$$= (-2)^{xy} \sum_{t=o}^{y} \binom{y}{t} (-1)^{xt} 2^{-xt} T^t \mathcal{J}_x^t B^{xt+u}$$

$$= (-2)^{xy} \left\{ \sum_{t=o}^{\left\lfloor \frac{y}{2} \right\rfloor} \binom{y}{2t} 2^{-2xt} T^{2t} \mathcal{J}_x^{2t} B^{2xt+u} + \sum_{t=1}^{\left\lfloor \frac{(y-1)}{2} \right\rfloor} \binom{y}{2t+1} (-1)^{x(2t+1)} 2^{-x(2t+1)} T^{(2t+1)} \mathcal{J}_x^{(2t+1)} B^{x(2t+1)+u} \right\}$$

$$= (-2)^{xy} \left\{ \sum_{t=o}^{\left\lfloor \frac{y}{2} \right\rfloor} \binom{y}{2t} 4^{-xt} 9^t \mathcal{J}_x^{2t} B^{2xt+u} + \sum_{t=1}^{\left\lfloor \frac{(y-1)}{2} \right\rfloor} \binom{y}{2t+1} (-1)^x 2^{-x(2t+1)} 9^t T \mathcal{J}_x^{(2t+1)} B^{x(2t+1)+u} \right\}$$

Since, $B^{2xt+u} = \begin{pmatrix} \frac{\mathcal{L}_{2xt+u}}{2} & \frac{9\mathcal{J}_{2xt+u}}{2} \\ \frac{\mathcal{J}_{2xt+u}}{2} & \frac{\mathcal{L}_{2xt+u}}{2} \end{pmatrix}$

Hence the proof.

Corollary 2.5.1
For every $y \in N \cup \{0\}$ and $x, u \in Z$

$$\mathcal{L}_{2xy+u} \equiv (-1)^{xy} (2)^{xy} \mathcal{L}_u (\mathrm{mod}\ \mathcal{J}_x) \qquad (8)$$

$$\mathcal{J}_{2xy+u} \equiv (-1)^{xy} (2)^{xy} \mathcal{J}_u (\mathrm{mod}\ \mathcal{J}_x) \qquad (9)$$

3. Properties of Jacobsthal and Jacobsthal Lucas numbers using divisibility

Theorem 3.1
Let $x, y \in N$ and $x \geq 2$. Then $\mathcal{L}_x \mid \mathcal{L}_y$ if and only if $x \mid y$ and $\frac{y}{x}$ is an odd integer.

Proof
Let us assume that $\mathcal{L}_x \mid \mathcal{L}_y$ and $x \nmid y$
By Division algorithm $y = xq + r$ where $0 \leq r < x$

Paper ID: SR24612143018 DOI: https://dx.doi.org/10.21275/SR24612143018

If q is an even integer, then $q = 2a$ for some $a \in Z$
From (6), it is pointed out by

$$\mathcal{L}_y = \mathcal{L}_{2xy+r} \equiv (-1)^{(x+1)y} 2^{xa} \ \mathcal{L}_r (\bmod \ \mathcal{L}_x)$$

The conditions $\mathcal{L}_x \mid \mathcal{L}_y$ and g.c.d $(\mathcal{L}_x, 2^{mx}) = 1$ together gives $\mathcal{L}_x \mid \mathcal{L}_r$

This is a contradiction to the fact that $\mathcal{L}_r \nmid \mathcal{L}_x$ as $r < x$

Hence, our assumption that q is an even integer is wrong.

Therefore q is an odd integer and let it be $q = 2a + 1$

Thus $\mathcal{L}_y = \mathcal{L}_{2xa+x+r} \equiv (-1)^{(x+1)a} 2^{xa} \ \mathcal{L}_{x+r} (\bmod \ \mathcal{L}_x)$

Also, since $\mathcal{L}_x \mid \mathcal{L}_y$, $\mathcal{L}_x \mid 2^{mx} \mathcal{L}_{x+r}$

To prove $r = 0$

Suppose $r > 0$
By employing the identity $\mathcal{L}_{x+r} = 2\mathcal{L}_x \mathcal{J}_{r-1} + \mathcal{J}_r \mathcal{L}_{x+1}$, it is attained by $\mathcal{L}_x \mid \mathcal{J}_r \mathcal{L}_{x+1}$

But $g.c.d(\mathcal{L}_x, \mathcal{L}_{x+1}) = 1$ leads to $\mathcal{L}_x \mid \mathcal{J}_r$

This violates the conditions that $\mathcal{J}_r \leq \mathcal{J}_x \leq \mathcal{L}_x$
Therefore, $r = 0$ and $y = xq$ where q is an odd integer.

Conversely, suppose $x \mid y$ and $y = x(2a + 1)$ for some $a \in Z$

Then, $\mathcal{L}_y = \mathcal{L}_{2xa+x} \equiv (-1)^{(x+1)a} 2^{xa} \ \mathcal{L}_x (\bmod \ \mathcal{L}_x)$

Thus, $\mathcal{L}_x \mid \mathcal{L}_y$

Theorem 3.2
Let $x, y \in N$ and $x \geq 2$. Then $\mathcal{L}_x \mid \mathcal{J}_y$ if and only if $x \mid y$ and $\frac{y}{x}$ is an even integer.

Proof
Suppose $\mathcal{L}_x \mid \mathcal{J}_y$

If $x \nmid y$, then by division algorithm $y = xq + r$, $0 \leq r < x$ and $x \geq 2$.

If q is an odd integer, then $q = 2a + 1$ for some $a \in Z$

$$\mathcal{J}_y = \mathcal{J}_{x(2a+1)+r} \equiv (-1)^{(x+1)a} 2^{xa} \ \mathcal{J}_{x+r} (\bmod \ \mathcal{L}_x)$$

Thus, $\mathcal{L}_x \mid \mathcal{J}_{x+r}$ and hence $\mathcal{L}_x \mid 9 \mathcal{J}_{x+r}$

Because of the identity $9\mathcal{J}_{x+r} = \mathcal{L}_x \mathcal{L}_{x+1} + 2\mathcal{L}_{x-1} \mathcal{L}_r$, it is received that $\mathcal{L}_x \mid 2\mathcal{L}_{x-1} \mathcal{L}_r$

Since $g.c.d \ (\mathcal{L}_x, \mathcal{L}_{x-1}) = 1$, it is obtained by

$$\mathcal{L}_x \mid \mathcal{L}_r \text{ and } \mathcal{L}_x \leq \mathcal{L}_r$$

This is illogical to fact that $r < x$ gives $\mathcal{L}_r \mid \mathcal{L}_x$

Therefore q is an even integer it is taken as $q = 2a$ for some $a \in Z$.

Then, from (7)

$$\mathcal{J}_y = \mathcal{J}_{2xa+r} \equiv (-1)^{(x+1)a} 2^{xa} \ \mathcal{J}_r (\bmod \ \mathcal{L}_x)$$

Now $\mathcal{L}_x \mid \mathcal{J}_y$ implies that $\mathcal{L}_x \mid \mathcal{J}_r$.

But this is impossible to the statement that $r < x$ implies that $\mathcal{J}_r \leq \mathcal{J}_x < \mathcal{L}_x$.

Then it follows that $r = 0$ and $y = xq$ where q is an even integer.

Conversely, suppose $x \mid y$ and $y = 2xa$ for some $a \in Z$

Then, $\mathcal{J}_y = \mathcal{J}_{2xa} \equiv (-1)^{(x+1)a} (2)^{xa} \ \mathcal{J}_0 (\bmod \ \mathcal{L}_x)$

Consequently $\mathcal{L}_x \mid \mathcal{J}_y$

Theorem 3.3
Let $x, y \in N$ and $x \geq 3$. Then $\mathcal{J}_x \mid \mathcal{J}_y$ if and only if $x \mid y$

Proof
Assume that $\mathcal{J}_x \mid \mathcal{J}_y$, but $x \nmid y$
Then, $y = xq + r$ with $0 < r < x$.
Suppose that q is an even integer. Let us choose $q = 2a$ for some $a \in Z$

In the vision of (9), it is perceived that

$$\mathcal{J}_y = \mathcal{J}_{2xa+r} \equiv (-1)^{(x+1)a} 2^{xa} \ \mathcal{J}_r (\bmod \ \mathcal{J}_x)$$

But $\mathcal{J}_x \mid \mathcal{J}_y$ results that $\mathcal{J}_x \mid \mathcal{J}_r$.

This is not possible because $0 < r < x$ and $x \geq 3$ affords that $\mathcal{J}_r \mid \mathcal{J}_x$.

If q is an odd integer, then $q = 2a + 1$ for some $a \in Z$.

Thus, $\mathcal{J}_y = \mathcal{J}_{x(2a+1)+r} \equiv (-1)^{xa} \ \mathcal{J}_{x+r} (\bmod \ \mathcal{J}_x)$

From $\mathcal{J}_x \mid \mathcal{J}_y$, it follows that $\mathcal{J}_x \mid \mathcal{J}_{x+r}$

The identity $\mathcal{J}_{x+r} = \mathcal{J}_r \mathcal{J}_{x+1} + 2\mathcal{J}_{r-1} \mathcal{J}_x$ grades that $\mathcal{J}_x \mid \mathcal{J}_{x+1} \mathcal{J}_r$.

Now $g.c.d \ (\mathcal{J}_x, \mathcal{J}_{x+1}) = 1 \Longrightarrow \mathcal{J}_x \mid \mathcal{J}_r$

Again $\mathcal{J}_r \mid \mathcal{J}_x$, if $x \geq 3$ and $r < x$.

Hence, it tracks that $r = 0$ and $y = xq$. So $x \mid y$

Conversely if $x \mid y$, then $y = xq$ for some $q \in N$

Thus, $\mathcal{J}_y = \mathcal{J}_{xq} = \sum_{j=0}^{q} \binom{q}{j} \ \mathcal{J}_x^j \ \mathcal{J}_{x-1}^{q-j} \ \mathcal{J}_j \ 2^{q-j}$

Hence, it is clear that $\mathcal{J}_x \mid \mathcal{J}_y$.

Theorem 3.4
If $k \geq 2$ and t is an odd number, then $L_{2^k t} \equiv 2 \ (mod \ 5)$.

Proof

If t is an odd number, then $t \equiv \pm 1, \pm 3, \pm 5, \pm 7 \ (mod\ 8)$. Moreover, it can be proved by induction that $2^k \equiv 0, \pm 4 \ (mod\ 8)$ for $k \geq 2$.

Hence $2^k t \equiv 0, \pm 4 \ (mod\ 8)$.

Therefore, $2^k t = 8q$ or $2^k t = 8q \pm 4$ for $q \geq 0$. Then it seeks that

$Q_{2^k t} = Q_{8q} \equiv Q_0 \ (mod\ P_4)$ or $Q_{2^k t} = Q_{8q \pm 4} \equiv Q_{\pm 4} \ (mod\ P_4)$

Thus, $Q_{2^k t} \equiv 2{,}10 \ (mod\ 12), k \geq 2$.

4. Conclusion

In this manuscript, the congruence and divisibility properties of the Jacobsthal and Jacobsthal-Lucas sequences are investigated. By analysing in this manner, one can examine a wide range of relationships that connect any two sequences.

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