A Note on Fixed-Point Theorem for Nonlinear Mappings

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Abstract: In this paper we discussed the *generalization of the findings on fixed-point theorem in a complete convex metric space with uniformly normal structure of Mukherjee and Som as well as Gillespie and Williams.*

Keywords: Fix point theory, convex metric spaces, uniformly normal structure, Banach space.

1. Introduction

In Banach spaces numerous researchers have examined fixed point theorems in convex metric spaces. The current study establishes a fixed-point theorem in a complete convex metric space with uniformly normal structure. This extends the findings of Mukherjee and Som [3] as well as Gillespie and Williams [2].

Definition 1 A mapping $W: X \times X \times [0,1] \to X$ is considered a convex structure on metric space X if it satisfies the following condition for all $x, y \in X$ and $\lambda \in [0,1]$.

$$d(u, W(x, y, \lambda)) \leqslant \lambda d(u, x) + (1 - \lambda) d(u, y), \text{ for all } u \in X.$$

A metric space that has a convex structure is said to be a convex metric space.

Definition 2 A subset K of a convex metric space X is said to be convex if

$$W(x, y, \lambda) \in K \text{ for all } x, y \in K \text{ and } \lambda \in [0,1].$$

A Banach space is a complete convex metric space and each of its convex subsets is a convex metric space. Simple examples may be found in [4].

We will now discuss the concept of uniform normal structure in convex metric spaces. This idea is already familiar in the field of Banach spaces. References for further information on reflexivity in spaces with uniform normal structure can be found in Bae [1].

Definition 3 A convex metric space is said to have a uniformly normal structure if there is a number $h \in (0,1)$ such that, if K is bounded closed and convex subset of X, then their exist $x_0 \in K$ such that

$$\sup\{d(x_0, y): y \in K\} \leqslant h\delta(K)$$

where $\delta(K)$ is the diameter of the set K.

Theorem: Let complete convex metric space X with a uniformly normal structure and T be a mapping of a bounded closed and convex subset K of X into itself.

$$d(Tx, Ty) \leqslant ad(x, y) + b\{d(x, Tx) + d(y, Ty)\} + c\{d(y, Tx) + d(x, Ty)\},$$

where $a > 0, b > 0, c > 0$ with $3a + 2b + 4c \leqslant 1$ and $x, y \in K$. Then T has a unique fixed point in K.

Proof: Since X has uniformly normal structure, there exists an $h \in (0,1)$ such that

$$P = \{x \in K: d(x, Tx) \leqslant h\delta(K)\} \text{ is non empty.}$$

Let $F_1 =$ ClCo $[T(P)]$, the closed and convex hull of $T(P)$. We will show $F_1 \subset P$.

Let $z = F_1$ then any one of the following three cases may arise:

Case (1) $z \in T(P)$, this implies that $z = Tu, u \in P$. By the nature of T,

$$d(Tu, T(Tu)) \leqslant \frac{a+b+c}{1-b-c} h\delta(K) \leqslant \frac{a+b+c}{1-2a-b-3c} h\delta(K)$$

Using the fact that $3a + 2b + 4c \leqslant 1$, we get $d(Tu, T(Tu)) \leqslant h\delta(K)$, this implies that $z \in P$.

Case (2) $z \in Co[T(P)]$, then

$$d(z, Ty) \leqslant \left\{\frac{a+b+c}{1-a-2c}\right\}\{h\delta(K) + d(y, Ty)\}, \text{ for all } y \in K.$$

It follows that

$$d(z, Tz) \leqslant \left\{\frac{a+b+c}{1-a-2c}\right\}\{h\delta(K) + d(z, Tz)\}$$

$$i.e \quad d(z, Tz) \leqslant \left\{\frac{a+b+c}{1-2a-b-3c}\right\} h\delta(K).$$

Hence $z \in P$.

Case (3) z is limit point of $Co[(T(P)]$, this implies that there is a sequence (z_n) in $Co[T(P)]$ such that $z_n \to z$. Therefore $d(z, Tz) \leqslant d(z, z_n) + d(z_n, Tz)$. Equation (1) then implies

$$d(z,Tz) \leqslant \left\{\frac{a+b+c}{1-2a-b-3c}\right\} h\delta(K) + \frac{1-a-2c}{1-2a-b-3c} d(z,z_n).$$

Letting $n \to \infty$ we get,

$$d(z,Tz) \leqslant \left\{\frac{a+b+c}{1-2a-b-3c}\right\} h\delta(K),$$

Hence $z \in P$.

Now $F_1 \subset P$ implies that $TF_1 \subset TP \subseteq F_1$, which further implies that there exists (F_n), a decreasing sequence of closed and convex T-invariant subsets of K, such that $\delta(F_n) \leqslant h^n \delta(K)$ for each n. Hence T has a fixed point. The uniquencess of the fixed point is easy to see.

2. Summary

With the notion of convex metric space our theorem generalizes the result of Mukerjee and Som [3, theorem 1]. The result of Gillespie and Williams [2] can also be obtained as corollary of above theorem.

References

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