Convolution Structure of the Fractional Fourier-Laplace Transform

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Abstract: ***Fourier related transforms have innumerous application in variety of disciplines not only in engineering sides like signal processing, optics communication but also in music, economics etc. The Fractional Fourier-Laplace transform (FrFLT) which combines the properties of both Fractional Fourier transform (FrFT) and the Laplace transform is a powerful tool in the field of signal processing and mathematical physics, which provides a better understanding of the time-frequency analysis of signals. Also convolution is a powerful way of characterizing the input output relationship of time invariant linear system. Convolution has many applications in image processing, optics, digital data processing, statistics, physics, electrical engineering, probability theory, Fractional calculus etc. In this paper, we propose the convolution structure for the Fractional Fourier-Laplace transform (FrFLT) and prove its convolution theorem.***

Keywords: Fourier transform, Laplace transform, Fractional Fourier transform, Fractional Fourier-Laplace transform, signal processing

1. Introduction

Fourier-related transforms play a fundamental role in scientific, engineering, and technological advancements, serving as powerful tools for analyzing and manipulating signals and data. The Fourier transform, known for its versatility and applicability, finds extensive use in fields such as signal processing [1], optics [2], communication [3], music [4], economics [5], and geography [6].

Fourier transform provides the spectral content of a signal, but it fails when it comes to determining the time location of spectral components. This limitation is particularly problematic for analyzing signals that are non-stationary or time-varying. To address this issue, time-frequency representations are utilized to capture time-frequency patterns, which are crucial for the analysis of such signals [7].

The fractional Fourier transform (FrFT) was introduced by Victor Namias in 1980 [8]. Since then, it has become a prominent research topic due to its numerous benefits and advantages. The key advantage of the FrFT lies in its ability to effectively analyze time-varying signals that cannot be adequately handled using the classical Fourier transform. The FrFT finds applications in various fields, including quantum mechanics, quantum optics, optical systems, signal and image processing, communication, and the solution of differential equations [8, 9, 10]

The Laplace transform, a powerful integral transform introduced by Pierre-Simon Laplace in the late 18th century, is a key mathematical tool with broad applications in mathematics, physics, engineering, and probability theory [11, 12]. It converts a function of time into a function of a complex variable, often simplifying the analysis of linear time-invariant systems. Unlike the Fourier transform, which resolves a function or a signal into its modes of vibrations, the Laplace transformation resolves a function into its moments. The Laplace transform has found extensive use in solving differential equations, both ordinary and partial, by transforming them into algebraic equations that are more manageable [12, 13]. In physics and engineering, it is employed to analyze and design systems such as electrical circuits, control systems, and mechanical systems [14]. In signal processing, the Laplace transform simplifies the analysis of signals and systems in the time domain [15].

We focus on the fractional Fourier-Laplace transform (FrFLT) which combines the properties of both fractional Fourier transform (FrFT) and the Laplace transform [16]. This may allow for more comprehensive analysis of signals that vary over time and frequency. Hence FrFLT may have applications in various fields like engineering, medical science, physics etc.

Also, Convolution is a mathematical operation that combines two functions to produce a third function that represents how one function modifies the other [17]. It is widely used in various fields, including signal processing, image processing, and physics. For example, in signal processing, convolution is used for tasks such as filtering, modulation, and deconvolution [1, 18]. In image processing, convolution is used for tasks such as blurring, sharpening, and edge detection [19].

In this paper, we propose a convolution structure for Fractional Fourier-Laplace transform (FrFLT) and based on this structure we prove the convolution theorem for FrFLT.

2. Fractional Fourier-Laplace Transform

1) The Conventional Fractional Fourier Laplace Transform

The fractional Fourier-Laplace transform with parameter α and θ of $f(x,t)$ is denoted by $FrFLT\{f(x,t)\}$ performs a linear operation given by the integral transform

$FrFLT\{f(x,t)\} = F_{\alpha,\theta}(u,v)$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\; K_{\alpha,\theta}(x,u,t,v)dx\,dt \qquad (2.1)$$
where,
$K_{\alpha,\theta}(x,u,t,v) =$
$$\sqrt{\frac{1-i\cot\alpha}{2\pi}}\sqrt{\frac{1-i\cot\theta}{2\pi i}}\, e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha-2xu]} \times e^{\frac{1}{2\sin\theta}[(t^2+v^2)\cos\theta-2tv]}$$

Let denote $C(\alpha,\theta) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\sqrt{\frac{1-i\cot\theta}{2\pi i}}$, $a(\alpha) = \frac{\cot\alpha}{2}$, $b(\theta) = \frac{\cot\theta}{2}$, $p(\alpha) = \sec\alpha$ & $q(\theta) = \sec\theta$ then

$$K_{\alpha,\theta}(x,u,t,v) = Ce^{ia(x^2+u^2-2xup)}\, e^{b(t^2+v^2-2tvq)} \qquad (2.2)$$

2) The Testing Function space $E(\mathbb{R}^n)$

An infinitely differentiable complex valued smooth function $\emptyset$ on $\mathbb{R}^n$ belongs to $E(\mathbb{R}^n)$, if for each compact subset $K \subset S_a$, $I \subset S_b$,
where, $S_a = \{x \in \mathbb{R}^n : |x| \le a, a > 0\}$ and
$S_b = \{t \in \mathbb{R}^n : |t| \le b, b > 0\}$
$$\gamma_{E,l,q}(\emptyset) = \sup_{\substack{x\in K\\ t\in I}}\left|D_x^l D_q^t \emptyset(t,x)\right| < \infty, \qquad l,q = 0,1,2,\ldots$$
Thus $E(\mathbb{R}^n)$ will denote space of all $\emptyset \in E(\mathbb{R}^n)$ with compact support contained in $S_a \cap S_b$. Moreover, we say that f is a fractional Fourier-Laplace transformable if it is a member of E^*, the dual space of E.

3) Distributional Generalized Fractional Fourier-Laplace Transform

The distributional fractional Fourier-Laplace transform of $f(x,t) \in E(\mathbb{R}^n)$ is defined by
$$FrFLT\{f(x,t)\} = F_{\alpha,\theta}(u,v) = \langle f(x,t), K_{\alpha,\theta}(x,u,t,v)\,\rangle \qquad (2.3)$$
where, $K_{\alpha,\theta}(x,u,t,v)$ is as given in (2.2). The R.H.S. of (2.1) is meaningful because $K_{\alpha,\theta}(x,u,t,v) \in E$ and $f(x,t) \in E^*$.

3. Convolution Structure

If $f(x)$ and $g(x)$ are two functions with Fourier transforms $\mathcal{F}(f)$ and $\mathcal{F}(g)$ respectively, then the convolution of f and g is given by:
$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt \qquad (1.1)$$
The convolution of two two-dimensional functions $f(x,y)$ and $g(x,y)$ is defined in ([16] page 331). Similarly, we can define the convolution of two fractional Fourier-Laplace transformable functions $f(x,t)$ and $g(x,t)$ by
$$(f * g)(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r,s)g(x-r,t-s)drds \qquad (3.2)$$
The convolution theorem for the Fourier transform states that, the Fourier transform of the convolution of two functions is equal to the product of the Fourier transforms of the individual functions i.e.
$$\mathcal{F}(f * g) = \mathcal{F}(f)\,\mathcal{F}(g)$$
However, the corresponding theorem for the FrFLT appears to lack the same elegance and practicality. This discrepancy, in our view, stems from the convolution operation defined by equation (3.2) not being the suitable type of convolution for the FrFLT.

Zayed defined the convolution structure for the fractional Fourier transform (FrFT) in [20]. Similarly, we introduce the convolution structure for the fractional Fourier-Laplace transform (FrFLT). For this, we first define the function $\tilde{f}$ as follows:

Definition: For any function $f(x,y)$ we define $\tilde{f}(x,y)$ by $\tilde{f}(x,y) = f(x,y)e^{i(ax^2-iby^2)}$ where $a = a(\alpha)$ & $b = b(\theta)$ are as given in(2.2). Then for any two functions f and g, we define the convolution operation $\star$ by
$$h(x,t) = (f \star g)(x,t) = Ce^{-i(ax^2-ibt^2)}(\tilde{f} * \tilde{g})(x,t) \quad (3.3)$$
Where $*$ is the convolution operation given in (3.2) & $C = C(\alpha,\theta)$ as given in (2.2). This gives the convolution structure for FrFLT. See Fig. 1 for a realization of Convolution operation $\star$

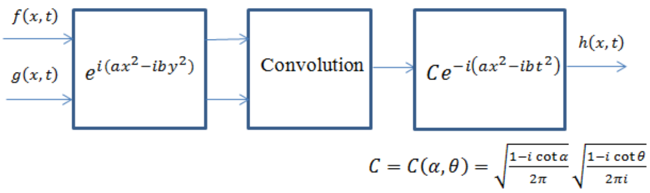

Fig. 1 Convolution for the FrFLT

4. Convolution Theorem for Fractional Fourier-Laplace Transform

Let $h(x,t) = (f \star g)(x,t)$ and $F_{\alpha,\theta}$, $G_{\alpha,\theta}$ & $H_{\alpha,\theta}$ denotes the fractional Fourier-Laplace transform (FrFLT) of f, g and h respectively. Then
$$H_{\alpha,\theta}(u,v) = F_{\alpha,\theta}(u,v)G_{\alpha,\theta}(u,v)\; e^{-i(au^2-ibv^2)} \quad (4.1)$$

Proof: From the definition of FrFLT we have,
$$H_{\alpha,\theta}(u,v) = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,t)e^{ia(x^2+u^2-2xup)}\; e^{b(t^2+v^2-2tvq)}dx\,dt$$
$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(f \star g)(x,t)e^{ia(x^2+u^2-2xup)} \times e^{b(t^2+v^2-2tvq)}dx\,dt$$
$$= C^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(ax^2-ibt^2)}(\tilde{f} * \tilde{g})(x,t) \times e^{ia(x^2+u^2-2xup)}e^{b(t^2+v^2-2tvq)}dx\,dt$$
$$= C^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(ax^2-ibt^2)}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(r,s) \times \tilde{g}(x-r,t-s)drds\right\}$$

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$$\times e^{ia(x^2+u^2-2xup)}\, e^{b(t^2+v^2-2tvq)}\, dxdt$$

$$= C^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(ax^2-ibt^2)} \Big\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r,s)$$

$$\times e^{i(ar^2-ibs^2)} g(x-r, t-s) e^{i\left(a(x-r)^2-ib(t-s)^2\right)} drds \Big\}$$

$$\times e^{ia(x^2+u^2-2xup)}\, e^{b(t^2+v^2-2tvq)}\, dxdt$$

Substituting $x-r=m, t-s=n$

$\Rightarrow x=m+r \,\&\, t=n+s$ then,

$H_{\alpha,\theta}(u,v) =$

$$C^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r,s) g(m,n)$$

$$\times e^{-i\left(a(m+r)^2-ib(n+s)^2\right)} e^{i(ar^2-ibs^2)} e^{i(am^2-ibn^2)}$$

$$\times e^{ia((m+r)^2+u^2-2(m+r)up)}$$

$$\times e^{b((n+s)^2+v^2-2(n+s)vq)} drdsdmdn$$

$$= C^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r,s) g(m,n)$$

$$\times e^{-i\left(am^2+ar^2+2amr-i(bn^2+bs^2+2bns)\right)} e^{i(ar^2-ibs^2)}$$

$$\times e^{i(am^2-ibn^2)} e^{i(am^2+ar^2+2amr+au^2-2amup-2arup)}$$

$$\times e^{(bn^2+bs^2+2bns+bv^2-2bnvq-2bsvq)} drdsdmdn$$

$$= C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r,s) e^{ia(r^2+u^2-2rup)}\, e^{b(s^2+v^2-2svq)} dr\, ds$$

$$\times C e^{-iau^2} e^{-bv^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(m,n) e^{ia(m^2+u^2-2mup)}$$

$$\times e^{b(n^2+v^2-2nvq)} dm\, dn$$

$$= e^{-iau^2} e^{-bv^2} C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r,s) e^{ia(r^2+u^2-2rup)}$$

$$\times e^{b(s^2+v^2-2svq)} dr\, ds$$

$$\times C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(m,n) e^{ia(m^2+u^2-2mup)}$$

$$\times e^{b(n^2+v^2-2nvq)} dm\, dn$$

$$= e^{-i(au^2-ibv^2)} F_{\alpha,\theta}(u,v) G_{\alpha,\theta}(u,v)$$

This completes the proof.

5. Conclusion

In this paper, we have proposed the convolution structure for Fractional Fourier-Laplace transform (FrFLT) and proved its convolution theorem. This may have applications in signal processing, mathematical physics, image processing & related fields. Additionally, we have presented generalization of FrFLT in a distributional sense.

Compliance with Ethical Statement

The authors confirm that there are no ethical issues related to this study as it involves purely mathematical research with no human or animal subjects.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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Ethical Conduct

This study did not involve any human or animal subjects, and therefore ethical approval was not required.

Data Availability Statement

The data and information supporting this research are available in the referenced research papers, which have been cited in the manuscript.

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