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On Connected Concave Fuzzy Sets

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Abstract: ***This article aims to study the connectedness of concave fuzzy sets with the help of fuzzy epigraph. Some results on concave fuzzy sets, α−level sets and support of fuzzy sets are discussed. relation between concave fuzzy sets and monotonic increasing and monotonic decreasing intervals on the domain R is established.***

Keywords: Fuzzy sets, Concave Fuzzy sets, Concave Fuzzy Relations, α−level sets, Fuzzy Epigraph

1. Introduction

In contrast to convexity, concavity is a fundamental concept that is examined in many concepts in pure and practical research. These function′s concavity facilitates the investigation and feature isolation of their characteristics. The terms fuzzy set, convex fuzzy set, fuzzy relation, and α-cut were initially introduced by Prof. Zadeh in 1965. Fuzzy set B is a function B:R→[0, 1], termed a membership function. B(x) is the membership grade at x in B; x∈R; values of membership grades lie in [0,1]. A fuzzy set defined on the Cartesian product of the set of real numbers is called a fuzzy relation M. It was during his research into several varieties of fuzzy sets that B.B. Choudhuri [10] first conceptualised concave fuzzy sets.Convex and concave fuzzy mappings were added to the notion by Yu-Ru Syau [11]. Sarkar [12] not only introduced concavo-convex fuzzy sets but also illustrated some other intriguing characteristics of this particular kind of fuzzy set. Ban constructed and thoughtfully explored convex temporal intuitionistic fuzzy sets as well as convex intuitionistic fuzzy sets [13, 14]. The generalised features of the aggregation of convex intuitionistic fuzzy sets were thoroughly analysed and characterised by Díiaz et al. [15].

Scholars Syau [5] and Xinmin Yang [2] demonstrated closed and convex fuzzy sets and investigated how they related to one another. In their study, Nadaban and Dzitac [4] discriminated between several forms of fuzzy relations and also gave examples of convex fuzzy relations. Chen-Wei-Xu [6] produced novel fuzzy relations and convexity results for fuzzy relations based on earlier work. Circular and elliptic holes are represented as concave fuzzy sets in [10], which defines and illustrates complementary α-level sets of fuzzy sets. If the membership grade of point t on the line segment connecting p and q is less than or equal to the maximum of the membership grade of $p \ and \ q \ \forall \ p, q \in R^n$, then the fuzzy set F is concave. Its qualities will be examined in this study. We aim to demonstrate that if a fuzzy relation T is concave, then its complementary β-level set, M^{β} is convex; $\forall \beta \in (0, 1]$. We will do this by extending the concavity of a fuzzy set with respect to complementary β-level set to fuzzy relations. We will examine the relationship between a Connected concave fuzzy set and relation between concave fuzzy set and convex fuzzy set is discussed.

2. Preliminaries

Throughout this paper, B denotes fuzzy set defined on M denotes fuzzy relation defined on R^2 Here are some definitions that will be useful in this paper.

2.1 Definition [6]:

A fuzzy set B defined on R is a function; $B: R \to [0,1]$ is called as membership function and $B(x)$ is called membership grade of $B \ at \ x$.

2.2 Definition [7]:

A Fuzzy relation M is a fuzzy set defined on Cartesian product of crisp sets $Y_1 \times Y_2 \times Y_3 \times \dots . . \times Y_n$ where tuples $(y_1, y_2, y_3, \dots . . , y_n)$ that may have varying degrees of membership value is usually represented by a real number for closed intervals[0,1] and indicate the strength of the present relation between elements of the topic. Consider $M: X \times Y \to [0,1]$ then the fuzzy relation on $X \times Y$ denoted by M or $M(x,y)$is defined as the set $M(X,Y) = \{((x,y), S(x,y))/(x,y) \in X \times Y\}$. where $M(x,y)$ is the strength of the relation in two variables called membership function. It gives the degree of membership of the ordered pair $(x,y) \ in \ X \times Y$ a real number in the interval [0,1].

2.3 Definition [3]:

M be fuzzy relation on $X \times Y$. Then T is concave if and only if $M(\mu(x_1, y_1) + (1-\mu)(x_2, y_2)) \leq Max[M(x_1, y_1) \wedge M(x_2, y_2)]$; $\forall \ (x_1, y_1), (x_2, y_2) \in X \times Y \ and \ \mu \in [0,1]$.

2.4 Definition[6]:

Let M be a fuzzy relation defined on $X \times Y$ and α be such that $0 < \beta \leq 1$.Then $Complementary \ \beta$ −level set of M is denoted by $M_{\beta}-$ is defined by $M_{\beta}-= \{(x,y) \in X \times Y / T(x,y) \leq \beta\}$.

2.5 Definition [1]:

B be a fuzzy set defined on R and α be such that $0 < \beta \leq 1$.Then β −level of B, is denoted by B_{β} and defined by $B_{\beta} = \{x \in R / B(x) \geq \beta\}$ is a crisp set.

2.6 Definition [1]:

B be a fuzzy set defined on R. Then M is concave if and only if $B(\mu x_1 + (1-\mu)x_2) \leq max[B(x_1), B(x_2)]$; $\forall\, x_1, x_2 \in R$ and $\mu \in (0, 1]$.

2.7 Definition[2]:

A fuzzy set B on R is said to be strongly convex fuzzy set if $B(\mu x_1 + (1-\mu)x_2) < max[B(x_1), B(x_2)]$; $\forall\, x_1, x_2 \in R, x_1 \neq x_2$ and $\mu \in (0,1)$.

2.8 Definition [10]:

A fuzzy set B on R is said to be a concave fuzzy set then B is connected if its complement is connected.

2.9 Definition [2]:

A fuzzy set B on is said to be strictly concave fuzzy set if $B(\mu x_1 + (1-\mu)x_2) < max[B(x_1), B(x_2)]$; $B(x_1) \neq B(x_2), \forall x_1, x_2 \in R$ and $\mu \in (0,1)$.

2.10 Fuzzy Epigraph

Fuzzy epigraph of a fuzzy set M is denoted by $f.epi(M) = \{(x,t): x \in R^n, t \in (0, M(x)]\}$.

2.11 Definition [1]:

B be a concave fuzzy set defined on R and α be such that $0 < \beta \leq 1$.Then complementary β −level of B^β, is denoted by $B^\beta_{\square}$ and defined by $B^\beta_{\square} = \{x \in R / B(x) \leq \beta\}$ is a crisp set.

2.12 Definition [16]:

A support of a fuzzy set B is the set $Supp(B) = \{x \in R / B(x) > 0\}$. It is obvious that S(B) = 0 iff B = 0.

3. Main Results

3.1 Theorem

B be a fuzzy set defined on R then B is Connected concave fuzzy set if and only if its fuzzy epigraph; $f.epi(B) = \{(x,t): x \in R^n, t \in (0, B(x)]\}$ is connected.

Proof:
Suppose M is a Connected concave fuzzy set defined on R. to prove that fuzzy epigraph; $f.epi(M) = \{(x,t): x \in R^n, t \in (0, M(x)]\}$ is connected.

Let, if possible, fuzzy epigraph; $f.epi(M) = \{(x,t): x \in R^n, t \in (0, M(x)]\}$ is not connected.

Then there exist two non-empty, disjoint, open sets X and Y such that $f.epi(M) = X \cup Y$ and $X \cap Y = \phi$.

We can choose $(x,y) \in X$ and $(x', y') \in Y$ and $(x,y) \in \notin X$ and $(x', y') \notin X$.
Here
y and y' are membership grades of x and x' respectively.

Without loss of generality, $y < y'$.

Consider $y' -$ level set of B; $B^{y'}$ is the union of two disjoint, open sets say S and T then $B^{y'}$ is not connected.

By definition of complementary β −level set of fuzzy set,

Concave fuzzy set is not connected.

A contradiction to our assumption that $B_{\square}$ is connected concave fuzzy set.

Therefore $f.epi(M) = \{(x,t): x \in R^n, t \in (0, M(x)]\}$ is connected.

Conversely suppose that $f.epi(M) = \{(x,t): x \in R^n, t \in (0, M(x)]\}$ is connected.

To prove that $B_{\square}$ is connected concave fuzzy set.

That is to prove that complementary β −level set of the concave fuzzy set is connected.

Let it possible for some η, B^η is not connected; $0 < \eta \leq 1$.

Then there exist two disjoint, nonempty, open sets A and B such that $B^\eta = A \cup B$ and $A \cap B = \phi$.
Construct hyperplane such that all the point of set $\{(x,t): x \in A$ and $t \in (0,1]\}$ and set $\{(y,t'): y \in B$ and $t' \in (0,1]\}$.
Gives $f.epi(M) = \{(x,t): x \in A$ and $t \in (0,1]\} \cup \{(y,t'): y \in B$ and $t' \in (0,1]\}$ and $\{(x,t): x \in A$ and $t \in (0,1]\} \cap \{(y,t'): y \in B$ and $t' \in (0,1]\} = \phi$.

Contradiction to the assumption that $f.epi(M)$ is connected subset of $R^n \times (0,1]$.

Therefore, B is connected concave fuzzy set defined on R.

3.2 Corollary

B be a strongly (strictly) concave fuzzy set defined on R then B^β is not convex; for all $\alpha \in (0, 1]$, where B^β is strong complementary β −cut of B.

3.3 Theorem

Concave fuzzy set B can be written as a union of all its complementary $\beta -$ level sets; $\forall\, \beta \in (0,1]$ i.e.
$B = \cup_{\beta \in (0,1]}^{\square} \square B^\beta$.

Proof.

Let $x \in B$.

To prove that $x \in \cup_{\beta \in (0,1]}^{\square} \square B^\beta$; for some $\beta \in (0,1]$.
Take $\beta = 1$ and consider complementary $\beta -$ level set.
Then B^1 is the largest complementary $\beta -$ level set containing x.
Therefore $x \in \cup_{\beta \in (0,1]}^{\square} \square B^\beta$.

Conversely, suppose that $x \in \cup_{\beta\in(0,1]} B^{\beta}$.

To prove that $x \in B$.

Without loss of generality assume that $x \in B^{\lambda}$ $for\ some\ \lambda \in (0,1]$.

Then $B(x) \le \lambda$. therefore, $(x, B(x)) \in B$.

Hence, we can write concave fuzzy set B is the a union of all its complementary $\beta - level\ sets; \forall\ \beta \in (0,1]$.

3.4 Corollary

Strongly Concave fuzzy set B can be written as a union of all its complementary $Strongly\ \beta - level\ sets; \forall\ \beta \in (0,1)\ i.e.$
$B = \cup_{\beta\in(0,1)} B^{\beta}$.

3.4 Theorem

B be a concave fuzzy set defined on
$R\ \ if\ and\ only\ if\ there\ is\ an\ interval\ I^{-}\ on\ which\ B\ is\ decre$
There is an interval I^{+} on which B is incresing.

Proof.
Let B be a concave fuzzy set defined on R.
$x^{1}\ , x^{2}\ \in I^{-}\ \subseteq B\ be\ such\ that\ x^{1}\ >\ x^{2}$.
Now to prove that $B(x^{1}) <\ B(\ x^{2})$.

Assume the contrary that $B(x^{1}) >\ B(\ x^{2})$

Since B is a concave fuzzy set defined on R.

We have,
Consider, $B(\eta x^{1}\ + (1-\eta)x^{2}) \le max[B(x^{1}), B(x^{2})]$$=B(x^{1})$; $.\eta \in [0,1)$.
For $\eta = 0$, we have $B(x^{1}) \le\ B(\ x^{2})$.

Contradiction to our assumption.
Therefore, $B(x^{1}) <\ B(\ x^{2})$.

In the same manner one can show that there is an interval I^{+} on which B is increasing.

Conversely, suppose that
$R\ \ if\ and\ only\ if\ there\ is\ an\ interval\ I^{-}\ on\ which\ B\ is\ decreasing$
There is an interval I^{+} on which B is incresing.

To prove that B is a concave fuzzy set defined on R..
Let $x^{1}, x^{2}\ \in I^{+}\ \subseteq B$ and $x^{1}\ >\ x^{2}$

Consider, $B(\eta x^{1}\ + (1-\eta)x^{2}) \le B(x^{1}) = max\ [B(x^{1}), B(x^{2})]; \eta \in [0,1)$.

Therefore, B is a concave fuzzy set defined on R.

3.5 Theorem

1) If B is a concave fuzzy set then Supp (B) is a convex set.
2) If B is a strongly Concave fuzzy set then Supp(B) $=R$.

Proof
To prove that Supp(B) is convex set.
Let $x^{1}\ , x^{2}\ \in Supp(B)$.

To prove that $\eta x^{1}\ + (1-\eta)x^{2}\ \in Supp(B)\ for\ some\ \ \eta \in [0,1)$.
Consider, $B(\eta x^{1}\ + (1-\eta)x^{2}) \le max[B(x^{1}), B(x^{2})]$.

Since, $x^{1}\ , x^{2}\ \in Supp(B)$ implies that $B(x^{1})$>0 and $B(x^{2})$>0.

Therefore, $\eta x^{1}\ + (1-\eta)x^{2}\ \in Supp(B)$.

Now to prove that if B is a strongly Concave fuzzy set then Supp(B) $=R$.
Let $x \in Supp(B)\ \ and\ Supp(B) \subseteq R\ \ implies\ x \in R$.

Therefore, $Supp(B) \subseteq R$.

Conversely, suppose that, $x \in R\ be\ any\ real\ number\ and\ B\ is\ strongly\ concave\ fuzzy\ set.$

Therefore, $B(\eta x^{1}\ + (1-\eta)x^{2}) \le max[B(x^{1}), B(x^{2})]$.

$$B(x) = B[\frac{1}{2}(x-y) + \frac{1}{2}B(x+y)]$$
$$\le max(B(x-y), B(x+y)) \ge 0.$$

$Therefore, x \in Supp(B). implies\ that\ \ R \subseteq Supp(B)$.

Supp(B) $=R$.

4. Conclusion

Fuzzy epigraphs and complementary α-level sets have been used to study the connectivity of concave fuzzy sets and concave fuzzy relations. It was demonstrated how the concave fuzzy set (concave fuzzy relation) and the fuzzy epigraph relate to one another. The complementary α-level set acts as a connection between fuzzy and crisp sets. The vast uses of concavity in various disciplines make a fuzzy approach to its research imperative on numerous levels.

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