Understanding Binary Operations and Algebraic Structures: A Foundational Approach to Abstract Algebra

Bobbili Hemalatha

Assistant Professor, Department of Mathematics, Nishitha Degree College (Autonomous)
Affiliated to Telangana University, Nizamabad
Accredited with "A" Grade By NAAC
Email: *hemalathab71[at]gmail.com*

Abstract: ***This paper explores the fundamental concepts of binary operations and algebraic structures, essential components of abstract algebra. Binary operations are defined as mappings on a set G.G and their properties, including associativity and commutativity, are discussed. The study examines algebraic structures such as groups, semi-groups, mono-ids, and groups, emphasizing their defining characteristics, examples, and applications. Key principles like identity and inverse elements, cancellation laws, and the conditions for a semi group to become a group are also addressed. By analyzing these structures, the paper highlights the systematic approach of abstract algebra in understanding the intrinsic properties of mathematical systems.***

Keywords: binary operations, algebraic structures, abstract algebra, associative operations, group theory

1. Introduction

Binary Operations and Algebraic Structures

- **Binary Operations:** Let G be a non-empty set, any mapping or function from GxG to G itself is called Binary operation on a set G i.e a function f: GxG→ G is a Binary operation on set G
- **Number of Binary Operations:** Number of binary operations on a non-empty finite set G with cardinality n is n^{n^2}
- **Algebraic Structure**: A non-empty set equipped with one or more binary operations is called Algebraic Structure. The algebraic structure consists of a set G and binary operations **∗, o** on G is denoted by **(G, ∗, o).**

Example of Binary Operation and Algebraic Structure

1) Let us consider G=N (Set of natural numbers) if ∀ a , b ∈ N , we define "**∗**" on N as
   a) **a∗b=a+b** then **(N, +)** is algebraic structure since sum of two natural numbers is again natural number.
   b) **a∗b=min{a,b}** , then **(N,∗)** is algebraic structure.
   c) **a∗b=a-b** then "**∗**" is not binary operation on N as 2 & 1 ∈ N but 1-2=-1∉ N

2) For G=R (Set of real numbers) if ∀ a , b ∈ R , we define "**∗**" on R as
   a) **a∗b=a+b** then **(R,+)** is algebraic structure
   b) **a∗b=** $\boldsymbol{a^b}$ then "**∗**" is not binary operation on G as for a=-1 and b= $\frac{1}{2}$

3) For G=Z (Set of integers) for every a , b ∈ Z , we define "**∗**" as
   a) **a∗b=a+b** then **(Z,+)** is algebraic structure
   b) **a∗b=ab**= ordinary multiplication of integer then **(Z,.)** is an algebraic structure.

2. Research

Associative Binary Operation

Let **∗** be a binary operation on a set G, then if two or more elements operated together, we get a string. A binary operation is said to be associative if any where in the string brackets are inserted then the result is unchanged. It is property of binary operation alone not of set, that is, a binary operation **∗** is called Associative if ∀ a, b, c ∈ **G, a∗(b∗c)=(a∗b)∗c.**

Commutative Binary Operation

Let **∗** be a binary operation on a set G, then **∗** is said to be commutative if and if **a∗b=b∗a**∀ a, b ∈ **G** i.e. each element of G commutes with each other.

Number of Commutative Binary Operation

Under a commutative binary operation on a set G, ordered pairs (a, b) and (b, a) are mapped to the same element so the total number of commutative operations on G of cardinality n is equal to

and the number of non-commutative binary operation= n- $\mathbf{n^2}$

Quasi Group or Groupoid

A non-empty set equipped with unique binary operations is called Quasi Group or Groupoid
Example: (N, +), (Z, +), (a, .) are called Groupoid

Semi Group

A Quasi group in which binary operation is associative is called Semi Group Example: (N, +) ,(N, .), (Q, .), (Q,+), (R, .), (Z, +), (Z, .) all are Semi Group

Identity Element

Let G be a non-empty set and **∗** be a binary operation on G then the element **e** ∈ **G** such that **a ∗e=e ∗a=a** ∀ a ∈G then "**e**" is called identity element with respect to **∗**

Monoid

A semi group (G, ∗) is said to be a monoid if it has an identity element. Example:

- (N, .) is monoid with identity element 1
- (Z, +) is monoid with identity element 0

Inverse Element

Let ∗ be a binary operation on a set G and let e be the identity element in G for binary operation ∗ then the element $a^{-1} \in G$ is said to be an inverse of
a∈ S if a ∗ $a^{-1} = a^{-1}$ ∗ a= e

Example:

- In (Z,.) only 1 and -1 have inverse elements
- In (R, +) every element has an inverse
- In (R, .) every non-zero element has an inverse

Group

A monoid in which each element has inverse, that is A non-empty set G with a binary operation ∗ is called a group if for every a, b, c ∈ G, the following properties hold:

- P1: (a ∗ b)∗ c=a∗ (b∗ c) called Associative Law
- P2: There exists e∈ G, such that a ∗e=e ∗a=a (Existence of identity element)
- P3: For each a ∈ G there exists -1 ∈ G such that a ∗−1 = ∗a=e(Existence of inverse)

Basic Properties of Group G

1) Cancellation Laws:
   a) Left Cancellation Law: If a, b, c∈ G then ab=ac implies b=c
   b) Right cancellation Law: If a, b, c∈ G then ba=ca implies b=c
2) A finite semi group G is a group if and only if G satisfies both the cancellation laws
3) The identity element of a group is unique
4) The inverse of each element in a group is unique

3. Conclusion

Abstract algebra is one of the divisions in algebra which come up with the truths relating to algebraic systems independent of the specific nature of some operations. These operations, in determined cases, have certain properties. Thus, we can come to an end with some termination of such assets. Therefore, this branch of mathematics called abstract algebra. Abstract algebra trade in with algebraic structures like the fields, groups, modules, rings, lattices, vector spaces, etc.

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