Operational Calculus for the Fractional Fourier-Laplace Transform

Vidya Sharma[1], Akash Patalwanshi[2]

[1]Professor and Head, Department of Mathematics, Smt. Narsamma Arts, Commerce and Science College, Kiran Nagar, Amravati-444606, Maharashtra, India
[2]Department of Mathematics, Smt. Narsamma Arts, Commerce and Science College, Kiran Nagar, Amravati-444606, Maharashtra, India
Corresponding Author Email: akashpatalwanshi7[at]gmail.com

Abstract: ***In this paper, we introduce the Fractional Fourier-Laplace Transform (FrFLT), rigorously defining its structure, extending it to all fractional parameters. A comprehensive operational calculus for the FrFLT is developed by proving its core properties, including linearity, shifting, scaling, differentiation, and modulation. These properties describe how the FrFLT interacts with linear combinations of functions, spatial shifts, partial derivatives, scaling transformations, and harmonic modulations. Additionally, we analyze the behavior of the FrFLT for various parameter values, providing insights into its structure and operational utility. These results demonstrate the FrFLT's versatility and potential for applications in signal processing, mathematical physics, and engineering. Furthermore, this research establishes a foundation for future exploration of the FrFLT in solving fractional differential equations and related problems.***

Keywords: Fractional Fourier transform, Fractional Laplace transform, Fractional Fourier-Laplace transform, Signal processing.

1. Introduction

The Fractional Fourier Transform (FRFT) serves as a generalization of the classical Fourier transform and has garnered significant interest over the past two decades due to its diverse applications in areas such as signal analysis, optics, and signal reconstruction [1, 2]. Moreover, it has also found utility in advanced fields such as pattern recognition, cryptography, and fractal signal processing, highlighting its importance in both theoretical and practical domains [3, 4, 5, 6]. The FRFT provides a framework for addressing certain mathematical problems that the traditional Fourier transform cannot efficiently handle [2]. The concept of FRFT was implicitly introduced by N. Wiener in 1929 [7] as a tool for solving specific ordinary and partial differential equations encountered in quantum mechanics. However, it was not until 1980 that V. Namias, seemingly unaware of Wiener's earlier work, formally defined the FRFT to address similar problems arising from classical quadratic Hamiltonians in quantum mechanics [8]. Subsequent refinements were made by McBride and Kerr [9].

Namias conceptualized the FRFT as an extension of the Fourier transform to fractional orders, in which the standard Fourier transform corresponds to an order of one and the identity transform to an order of zero. Importantly, the semi-group (additive) property was preserved, ensuring that applying the transform of order one-half twice would yield the ordinary Fourier transform. Mathematically, the FRFT constitutes a family of operators $\{\mathcal{F}_\alpha\}$, indexed by a parameter α, where $0 \le \alpha \le 1$. Specifically,

$$\mathcal{F}_0[f] = f,\ \mathcal{F}_1[f] = \hat{f}$$

where $\hat{f}$ represents the Fourier transform of f, and the composition satisfies $\mathcal{F}_\alpha \mathcal{F}_\beta = \mathcal{F}_{\alpha+\beta}$.

By employing a suitable scaling, the FRFT can be parameterized by an angular parameter θ, with $0 \le \theta \le 2\pi$. Under this framework, $\mathcal{F}_0$ remains the identity transformation, $\mathcal{F}_{\pi/2}$ represents the Fourier transform, and additional properties include $\mathcal{F}_\theta = \mathcal{F}_{\theta+2\pi}$. Explicitly,

$$\mathcal{F}_0[f] = f,\ \mathcal{F}_{\pi/2}[f] = \hat{f},\ \mathcal{F}_\pi[f(x)] = f(-x),\ \mathcal{F}_{2\pi}[f] = f.$$

Similarly, the Fractional Laplace Transform (FRLT) extends the classical Laplace transform to fractional orders and serves as an important tool in both theoretical and applied mathematics. Analogous to the FRFT, the FRLT generalizes its corresponding classical transform, enabling continuous interpolation between different orders through a parameter that governs its behaviour. Introduced within the framework of canonical transforms, the FRLT has been shown to be relevant to solving certain parabolic differential equations and to quantum mechanical problems involving repulsive oscillators. Moreover, the FRLT exhibits properties such as continuity with respect to its order parameter and adherence to the semi-group property, allowing compositions of fractional transforms to yield transforms of combined orders. The FRLT also reduces to the standard Laplace and Bargmann transforms for specific parameter values, ensuring consistency with classical cases [10].

Both of these transforms and similar transforms exhibit several key mathematical properties, including linearity, shifting, scaling, differentiation, and modulation, which underpin their utility in theoretical and applied contexts [8, 11, 12, 13, 14, 15, 16, 17, 18]. These properties describe how the transforms respond to basic operations such as linear combinations, spatial or parameter shifts, scaled transformations and frequency modulations, forming the foundation for their practical applications.

In this paper, we introduce the fractional Fourier-Laplace Transform (FrFLT), a novel transformation that combines essential features of the fractional Fourier transform (FRFT) and fractional Laplace transform (FRLT) [19]. While the FrFLT was previously defined in [19] only for the fractional parameters $\alpha, \theta \notin \{0, \pi/2, \pi\}$, we extend its definition to encompass all values of α and θ. Additionally, we develop a comprehensive operational calculus for the FrFLT by rigorously establishing its fundamental properties, including linearity, shifting, differentiation, scaling, and modulation.

The paper is organized as follows. In the next section, we provide the necessary definitions, including a modified definition of the fractional Laplace transform, the fractional Fourier-Laplace Transform (FrFLT), the testing function space, and the distributional generalized fractional Fourier-Laplace Transform. In Section 3, we investigate the behavior of the FrFLT for various parameter values. In Section 4, we rigorously derive the core properties of the FrFLT. Finally, in the last section, Section 5, we conclude with a discussion of our findings and their potential applications.

2. Preliminaries

I. The Fractional Fourier transform

The fractional Fourier transform (FRFT) with angle α of a function $f(x)$ is defined in [11], can be expressed as

$$\mathcal{F}_\alpha\{f(x)\}(u) = F_\alpha(u) = \int_{-\infty}^{\infty} f(x)\ K_\alpha(x,u)\,dx \qquad (2.1)$$

where,

$$K_\alpha(x,u) = \begin{cases} \sqrt{\dfrac{1 - i\cot\alpha}{2\pi}}\, e^{i\left[(x^2+u^2)\frac{\cot\alpha}{2} - xu\csc\alpha\right]}, & \text{if } \alpha \neq m\pi \\ \delta(x-u), & \text{if } \alpha = 2m\pi \\ \delta(x+u), & \text{if } \alpha = (2m-1)\pi \end{cases} \qquad (2.2)$$

is a transformational kernel with $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}$ and δ denotes the Dirac delta function.

II. Fractional Laplace transform

The fractional Laplace transform (FRLT) with angle ϕ of a function $f(t)$ is defined in [10]. We now extend the definition of fractional Laplace transform given in [10] to include the case where ϕ is integral multiple of π, which were previously undefined. In this paper, we use the notation θ instead of ϕ. This extension ensures that the FRLT framework is complete and facilitates its seamless application in defining the Fractional Fourier-Laplace transform. The extended definition of FRLT is as follows:

The fractional Laplace transform (FRLT) with angle θ of a function $f(t)$ is defined as

$$\mathcal{L}_\theta\{f(t)\}(u) = \mathcal{L}_\theta(u) = \int_{-\infty}^{\infty} f(t)\ K_\theta(t,v)\,dt \qquad (2.3)$$

where,

$$K_\theta(t,v) = \begin{cases} \sqrt{\dfrac{1 - i\cot\theta}{2\pi i}}\, e^{\left[(t^2+v^2)\frac{\cot\theta}{2} - tv\csc\theta\right]}, & \text{if } \theta \neq n\pi \\ \delta(t-v), & \text{if } \theta = 2n\pi \\ \delta(t+v), & \text{if } \theta = (2n-1)\pi \end{cases} \qquad (2.4)$$

is a transformational kernel with $\theta \in \mathbb{R}$, $n \in \mathbb{Z}$ and δ denotes the Dirac delta function.

This extended definition agrees with the definition given by Torre in [10] when $\theta \neq n\pi$ and it also include the case where θ is integral multiple of π. In accordance with this extended definition, the FRLT reduced to the identity transform at $\theta = 2n\pi$ and at it reduce to reflection transform $f(-v)$ at $\theta = (2n-1)\pi$.

III. The Conventional Fractional Fourier-Laplace Transform

The fractional Fourier-Laplace transform with angle parameters α and θ of $f(x,t)$ is a linear operator given by the integral transform,

$$\mathcal{FL}_{\alpha,\theta}\{f(x,t)\} = F_{\alpha,\theta}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\; K_{\alpha,\theta}(x,u,t,v)dx\,dt \tag{2.5}$$

where,

$$K_{\alpha,\theta}(x,u,t,v) = \begin{cases} C(\alpha,\theta)e^{i[a(\alpha)(x^2+u^2)-p(\alpha)xu]}e^{[b(\theta)(t^2+v^2)-q(\theta)tv]}, & \text{if } \alpha,\theta \notin \{k\pi : k \in \mathbb{Z}\} \\ C_1(\alpha)\delta(t-v)e^{i[a(\alpha)(x^2+u^2)-p(\alpha)xu]}, & \text{if } \alpha \neq m\pi \text{ and } \theta = 2n\pi \\ C_1(\alpha)\delta(t+v)e^{i[a(\alpha)(x^2+u^2)-p(\alpha)xu]}, & \text{if } \alpha \neq m\pi \text{ and } \theta = (2n-1)\pi \\ C_2(\theta)\delta(x-u)e^{[b(\theta)(t^2+v^2)-q(\theta)tv]}, & \text{if } \alpha = 2m\pi \text{ and } \theta \neq n\pi \\ C_2(\theta)\delta(x+u)e^{[b(\theta)(t^2+v^2)-q(\theta)tv]}, & \text{if } \alpha = (2m-1)\pi \text{ and } \theta \neq n\pi \\ \delta(x-u)\delta(t-v), & \text{if } \alpha,\theta \in \{2k\pi : k \in \mathbb{Z}\} \\ \delta(x-u)\delta(t+v), & \text{if } \alpha = 2m\pi \text{ and } \theta = (2n-1)\pi \\ \delta(x+u)\delta(t-v), & \text{if } \alpha = (2m-1)\pi \text{ and } \theta = 2n\pi \\ \delta(x+u)\delta(t+v), & \text{if } \alpha,\theta \in \{2(k-1)\pi : k \in \mathbb{Z}\} \end{cases} \tag{2.6}$$

is a transformation kernel with

$$C_1(\alpha) = \sqrt{\frac{1-i\cot\alpha}{2\pi}},\;\; C_2(\theta) = \sqrt{\frac{1-i\cot\theta}{2\pi i}},\;\; C(\alpha,\theta) = C_1(\alpha)C_2(\theta),$$

$$a(\alpha) = \frac{\cot\alpha}{2},\;\; b(\theta) = \frac{\cot\theta}{2},\;\; p(\alpha) = \csc\alpha,\;\; q(\theta) = \csc\theta$$

Where $\alpha,\theta \in \mathbb{R}$, $m,n \in \mathbb{Z}$ and δ denotes the Dirac delta function. For simplicity we may write C_1, C_2, C, a, b, p, and q instead of $C_1(\alpha)$, $C_2(\theta)$, $C(\alpha,\theta)$, $a(\alpha)$, $b(\theta)$, $p(\alpha)$ and $q(\theta)$.

IV. The Testing Function space $E(\mathbb{R}^n)$

An infinitely differentiable complex valued smooth function $\emptyset$ on $\mathbb{R}^n$ belongs to $E(\mathbb{R}^n)$, if for each compact subset $K \subset S_a$, $I \subset S_b$ where, $S_a = \{x \in \mathbb{R}^n : |x| \leq a, a > 0\}$, $S_b = \{t \in \mathbb{R}^n : |t| \leq b, b > 0\}$ such that,

$$\gamma_{E,l,q}(\emptyset) = \sup_{\substack{x\in K \\ t\in I}} \left|D_x^l D_q^t \emptyset(x,t)\right| < \infty, \qquad l,q = 0,1,2,\ldots \tag{2.7}$$

Thus $E(\mathbb{R}^n)$ will denote space of all $\emptyset \in E(\mathbb{R}^n)$ with compact support contained in $S_a \cap S_b$. Moreover, we say that f is a fractional Fourier-Laplace transformable if it is a member of E^*, the dual space of E.

V. Distributional Generalized Fractional Fourier-Laplace Transform

The distributional fractional Fourier-Laplace transform of $f(x,t) \in E(\mathbb{R}^n)$ is defined by

$$FrFLT\{f(x,t)\} = F_{\alpha,\theta}(u,v) = \langle f(x,t), K_{\alpha,\theta}(x,u,t,v)\,\rangle \tag{2.8}$$

where, $K_{\alpha,\theta}(x,u,t,v)$ is as given in (2.6). The R.H.S. of (2.8) is meaningful because $K_{\alpha,\theta}(x,u,t,v) \in E$ and $f(x,t) \in E^*$.

3. Analysis of FrFLT for different parametric values

The FrFLT of $f(x,t)$, for some special values of α & θ are given in the **Table *1***. Since the calculations are straightforward, only final results will be tabulated. Note that, here $f(x,t)$ belongs to space W of all integrable functions with property that $f(x,t) \in W$ if and only if $\mathcal{F}_\alpha[f(x,t)]_x, \mathcal{L}_\theta[f(x,t)]_t \in W$. Where, $\mathcal{F}_\alpha[f(x,t)]_x$ is fractional Fourier transform of $f(x,t)$ with respect to x at angle α and $\mathcal{L}_\theta[f(x,t)]_t$ denotes the fractional Laplace transform of $f(x,t)$ with respect to t at angle θ.

Table 1: FrFLT for different parametric values

Values of $\boldsymbol{\alpha, \theta}$	$\boldsymbol{K_{\alpha,\theta}(x,u,t,v)}$	$\boldsymbol{\mathcal{FL}_{\alpha,\theta}[f(x,v)] = F_{\alpha,\theta}(u,v)}$
$\alpha, \theta \in \{2k\pi : k \in \mathbb{Z}\}$	$\delta(x-u)\delta(t-v)$	$F_{\alpha,\theta}[f](u,v) = f(u,v)$
$\alpha, \theta \in \{2(k-1)\pi : k \in \mathbb{Z}\}$	$\delta(x+u)\delta(t+v)$	$F_{\pi,\pi}[f](u,v) = f(-u,-v)$
$\alpha = 2m\pi, \theta = (2n-1)\pi$	$\delta(x-u)\delta(t+v)$	$F_{0,\pi}[f](u,v) = f(u,-v)$
$\alpha = (2m-1)\pi, \theta = 2n\pi$	$\delta(x+u)\delta(t-v)$	$F_{\pi,0}[f](u,v) = f(-u,v)$
$\alpha = 2m\pi$ and $\theta \neq n\pi$	$C_2\delta(x-u)e^{[b(t^2+v^2)-qtv]}$	$F_{\alpha,\theta}[f](u,v) = \mathcal{L}_\theta[f(u,t)]_t(v)$
$\alpha = (2m-1)\pi$ and $\theta \neq n\pi$	$C_2\delta(x+u)e^{[b(t^2+v^2)-qtv]}$	$F_{\pi,\theta}[f](u,v) = \mathcal{L}_\theta[f(-u,t)]_t(v)$
$\alpha \neq m\pi$ and $\theta = 2n\pi$	$C_1\delta(t-v)e^{i[a(x^2+u^2)-pxu]}$	$F_{\alpha,\theta}[f](u,v) = \mathcal{F}_\alpha[f(x,v)]_x(u)$
$\alpha \neq m\pi$ and $\theta = (2n-1)\pi$	$C_1\delta(t+v)e^{i[a(x^2+u^2)-pxu]}$	$F_{\alpha,\theta}[f](u,v) = \mathcal{F}_\alpha[f(x,-v)]_x(u)$
$\alpha = \theta = \frac{\pi}{2}$	$\sqrt{\frac{1}{2\pi}}\sqrt{\frac{1}{2\pi i}}e^{-ixu}e^{-tv}$	$F_{\frac{\pi}{2},\frac{\pi}{2}}[f](u,v) = \sqrt{\frac{1}{2\pi}}\sqrt{\frac{1}{2\pi i}}FL\{f(x,t)\}(u,v)$

Where $FL\{f(x,t)\}(u,v)$ denotes Fourier-Laplace transform of $f(x,t)$ as defined in [18]. Based on the analysis in **Table 1**, we shall now focus on $F_{\alpha,\theta}$ for $\alpha, \theta \notin \{k\pi : k \in \mathbb{Z}\}$. In this case the kernel of FrFLT is:

$$K_{\alpha,\theta}(x,u,t,v) = C(\alpha,\theta)e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]} \quad (3.1)$$

that is,

$$K_{\alpha,\theta}(x,u,t,v) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\sqrt{\frac{1-i\cot\theta}{2\pi i}}\,e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha-2xu]}e^{\frac{1}{2\sin\theta}[(t^2+v^2)\cos\theta-2tv]}$$

4. Some properties of FrFLT

In this section, we derived various operational transform formulae for the FrFLT, including results related to linearity, shifting, scaling, differentiation and modulation.

4.1 Linearity property

$$\mathcal{FL}_{\alpha,\theta}\{c_1f(x,t)+c_2g(x,t)\} = c_1\mathcal{FL}_{\alpha,\theta}\{f(x,t)\} + c_2\mathcal{FL}_{\alpha,\theta}\{g(x,t)\} \quad (4.1)$$

Proof: From the definition of FrFLT we have,

$$\mathcal{FL}_{\alpha,\theta}\{c_1f(x,t)+c_2g(x,t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(c_1f(x,t)+c_2g(x,t))\,K_{\alpha,\theta}(x,u,t,v)dx\,dt$$

$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(c_1f(x,t)+c_2g(x,t))\,e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]}\,dx\,dt$$

$$= c_1C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,t)\,e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]}\,dx\,dt$$

$$+c_2C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,t)\,e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]}\,dx\,dt$$

$$= c_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,t)\,K_{\alpha,\theta}(x,u,t,v)\,dx\,dt + c_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,t)\,K_{\alpha,\theta}(x,u,t,v)\,dx\,dt$$

$$= c_1\mathcal{FL}_{\alpha,\theta}\{f(x,t)\} + c_2\mathcal{FL}_{\alpha,\theta}\{g(x,t)\}$$ ■

4.2 Modulated shifting property

$$\mathcal{FL}_{\alpha,\theta}\{e^{i(\omega x-i\psi t)}f(x,t)\}(u,v)=e^{2ia\left(u-\frac{\omega}{2p}\right)\frac{\omega}{p}}e^{2b\left(v-\frac{\psi}{2q}\right)\frac{\psi}{q}}\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u-\frac{\omega}{p},v-\frac{\psi}{q}\right) \quad (4.2)$$

Proof: By definition of FrFLT we have,

$$\mathcal{FL}_{\alpha,\theta}\{e^{i(\omega x-i\psi t)}f(x,t)\}(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\omega x-i\psi t)}f(x,t)\,K_{\alpha,\theta}(x,u,t,v)\,dx\,dt$$

$$=C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\omega x-i\psi t)}f(x,t)\,e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]}\,dx\,dt$$

$$=Ce^{2ia\left(u-\frac{\omega}{2p}\right)\frac{\omega}{p}}e^{2b\left(v-\frac{\psi}{q}\right)\frac{\psi}{q}}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,t)e^{i\left[a\left(x^2+\left(u-\frac{\omega}{p}\right)^2\right)-px\left(u-\frac{\omega}{p}\right)\right]}e^{\left[b\left(t^2+\left(v-\frac{\psi}{q}\right)^2\right)-qt\left(v-\frac{\psi}{q}\right)\right]}\,dx\,dt$$

$$=e^{2ia\left(u-\frac{\omega}{2p}\right)\frac{\omega}{p}}e^{2b\left(v-\frac{\psi}{2q}\right)\frac{\psi}{q}}\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u-\frac{\omega}{p},v-\frac{\psi}{q}\right) \quad \blacksquare$$

4.3 Shifting properties

I $$\mathcal{FL}_{\alpha,\theta}\{f(x-x_0,t)\}(u,v)=e^{i(ax_0^2-px_0u)}e^{i\left(u-\frac{ax_0}{p}\right)\frac{4a^2x_0}{p}}\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u-\frac{2ax_0}{p},v\right) \quad (4.3)$$

II $$\mathcal{FL}_{\alpha,\theta}\{f(x,t-t_0)\}(u,v)=e^{(bt_0^2-qt_0v)}e^{\left(v-\frac{bt_0}{q}\right)\frac{4b^2t_0}{q}}\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u,v-\frac{2bt_0}{q}\right) \quad (4.4)$$

III $$\mathcal{FL}_{\alpha,\theta}\{f(x-x_0,t-t_0)\}(u,v)=e^{i(ax_0^2-px_0u)}e^{(bt_0^2-qt_0v)}e^{i\left(u-\frac{ax_0}{p}\right)\frac{4a^2x_0}{p}}e^{\left(v-\frac{bt_0}{q}\right)\frac{4b^2t_0}{q}}\times\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u-\frac{2ax_0}{p},v-\frac{2bt_0}{q}\right) \quad (4.5)$$

Proof: By definition of FrFLT we have,

(I) $$\mathcal{FL}_{\alpha,\theta}\{f(x-x_0,t)\}(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x-x_0,t)\,K_{\alpha,\theta}(x,u,t,v)\,dx\,dt$$

Substituting $x-x_0=z\Rightarrow dx=dz$ we get,

$$\mathcal{FL}_{\alpha,\theta}\{f(x-x_0,t)\}(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(z,t)\,K_{\alpha,\theta}(z+x_0,u,t,v)\,dz\,dt$$

$$=C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(z,t)\;e^{i[a((z+x_0)^2+u^2)-p(z+x_0)u]}e^{[b(t^2+v^2)-qtv]}\,dz\,dt$$

$$=Ce^{i(ax_0^2-px_0u)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2iazx_0}f(z,t)\,e^{i[a(z^2+u^2)-pzu]}e^{[b(t^2+v^2)-qtv]}\,dz\,dt$$

$$=Ce^{i(ax_0^2-px_0u)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2iaxx_0}f(x,t)\,e^{i[a(x^2+u^2)-pxu]}\,e^{[b(t^2+v^2)-qtv]}\,dx\,dt$$

$$=e^{i(ax_0^2-px_0u)}\mathcal{FL}_{\alpha,\theta}\{e^{2iaxx_0}f(x,t)\}$$

$$=e^{i(ax_0^2-px_0u)}e^{i\left(u-\frac{ax_0}{p}\right)\frac{4a^2x_0}{p}}\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u-\frac{2ax_0}{p},v\right) \quad \text{By (4.2)}$$

(II) $$\mathcal{FL}_{\alpha,\theta}\{f(x,t-t_0)\}(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,t-t_0)\,K_{\alpha,\theta}(x,u,t,v)\,dx\,dt$$

Substituting $t-t_0=y\Rightarrow dt=dy$ we get, $e^{b(y^2+v^2-qyv)}$

$$\mathcal{FL}_{\alpha,\theta}\{f(x,t-t_0)\}(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,K_{\alpha,\theta}(x,u,y+t_0,v)\,dx\,dy$$

$$\therefore\ \mathcal{FL}_{\alpha,\theta}\{f(x,t-t_0)\}(u,v)=C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\;e^{i[a(x^2+u^2)-pxu]}e^{[b((y+t_0)^2+v^2)-q(y+t_0)v]}\,dx\,dy$$

$$=Ce^{(bt_0^2-qt_0v)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2byt_0}f(x,y)e^{i[a(x^2+u^2)-pxu]}\,e^{[b(y^2+v^2)-qyv]}\,dx\,dy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2btt_0}f(x,t)\,e^{i[a(x^2+u^2)-pxu]}\,e^{[b(t^2+v^2)-qtv]}\,dx\,dt$$

$$=e^{(bt_0^2-qt_0v)}\,\mathcal{FL}_{\alpha,\theta}\{e^{2btt_0}f(x,t)\}$$

Paper ID: SR25119122327 DOI: https://dx.doi.org/10.21275/SR25119122327

$$= e^{(bt_0^2-qt_0v)}e^{\left(v-\frac{bt_0}{q}\right)\frac{4b^2t_0}{q}}\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u,v-\frac{2bt_0}{q}\right) \quad \text{By (4.2)}$$

(III) $\mathcal{FL}_{\alpha,\theta}\{f(x-x_0,t-t_0)\}(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x-x_0,t-t_0)\,K_{\alpha,\theta}(x,u,t,v)\,dx\,dt$

Substituting $x-x_0=z$, $t-t_0=y$ and hence $dx=dz$ and $dt=dy$ we get,

$$\mathcal{FL}_{\alpha,\theta}\{f(x-x_0,t-t_0)\}(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(z,y)\,K_{\alpha,\theta}(z+x_0,u,y+t_0,v)\,dz\,dy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(z,y)\,e^{i[a((z+x_0)^2+u^2)-p(z+x_0)u]}e^{[b((y+t_0)^2+v^2)-q(y+t_0)v]}\,dz\,dy$$

$$=Ce^{i(ax_0^2-px_0u)}e^{(bt_0^2-qt_0v)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2(iazx_0+byt_0)}f(z,y)\,e^{i[a(z^2+u^2)-pzu]}e^{[b(y^2+v^2)-qyv]}\,dz\,dy$$

$$=Ce^{i(ax_0^2-px_0u)}e^{(bt_0^2-qt_0v)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2(iaxx_0+btt_0)}f(x,t)\,e^{i[a(x^2+u^2)-pxu]}\,e^{[b(t^2+v^2)-qtv]}\,dx\,dt$$

$$=e^{i(ax_0^2-px_0u)}e^{(bt_0^2-qt_0v)}\mathcal{FL}_{\alpha,\theta}\{e^{2(iaxx_0+btt_0)}f(x,t)\}$$

$$=e^{i(ax_0^2-px_0u)}e^{(bt_0^2-qt_0v)}e^{i\left(u-\frac{ax_0}{p}\right)\frac{4a^2x_0}{p}}e^{\left(v-\frac{bt_0}{q}\right)\frac{4b^2t_0}{q}}\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u-\frac{2ax_0}{p},v-\frac{2bt_0}{q}\right) \quad \text{By (4.2)}$$

■

4.4 Scaling property

For nonzero real constant r and s,

$$\mathcal{FL}_{\alpha,\theta}\{f(rx,st)\}(u,v)=\sqrt{\frac{1-i\cot\alpha}{r^2-i\cot\alpha}}\sqrt{\frac{1-i\cot\theta}{s^2-i\cot\theta}}e^{i\left[\frac{\cot\alpha}{2}u^2\left(1-\frac{\cos^2\alpha}{\cos^2\beta}\right)\right]}e^{\left[\frac{\cot\theta}{2}v^2\left(1-\frac{\cos^2\theta}{\cos^2\gamma}\right)\right]} \times\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u\frac{\sin\beta}{r\sin\alpha},v\frac{\sin\gamma}{s\sin\theta}\right) \tag{4.6}$$

where, $\beta=\arctan(r^2\tan\alpha)$ and $\gamma=\arctan(s^2\tan\theta)$

Proof: By definition of FrFLT we have,

$$\mathcal{FL}_{\alpha,\theta}\{f(rx,st)\}(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(rx,st)\;K_{\alpha,\theta}(x,u,t,v)\,dx\,dt$$

$$=C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(rx,st)\,e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]}\,dx\,dt$$

Substituting $y=rx$ and $z=st$

Hence $dy=rdx$ and $dz=sdt \Rightarrow dx=\frac{dy}{r}$ and $dt=\frac{dz}{s}$

$$\therefore \mathcal{FL}_{\alpha,\theta}\{f(rx,st)\}(u,v)=C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\left[a\left(\frac{y^2}{r^2}+u^2\right)-p\frac{y}{r}u\right]}e^{\left[b\left(\frac{z^2}{s^2}+v^2\right)-q\frac{z}{s}v\right]}f(y,z)\frac{dy}{r}\frac{dz}{s}$$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\sqrt{\frac{1-i\cot\theta}{2\pi i}}\frac{1}{|rs|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\left[\frac{\cot\alpha}{2r^2}(y^2+u^2r^2)-yu\frac{\csc\alpha}{r}\right]}e^{\left[\frac{\cot\theta}{2s^2}(z^2+v^2s^2)-zv\frac{\csc\theta}{s}\right]}f(y,z)\,dy\,dz$$

Now , if we define $\beta=\operatorname{arccot}\left(\frac{\cot\alpha}{r^2}\right)=\arctan(r^2\tan\alpha)$ and $\gamma=\operatorname{arccot}\left(\frac{\cot\theta}{s^2}\right)=\arctan(s^2\tan\theta)$ then we get,

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi r^2}}\sqrt{\frac{1-i\cot\theta}{2\pi s^2 i}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\left[\frac{\cot\beta}{2}(y^2+u^2r^2)-yu\frac{\csc\alpha}{r\csc\beta}\csc\beta\right]}e^{\left[\frac{\cot\gamma}{2}(z^2+v^2s^2)-zv\frac{\csc\theta}{s\csc\gamma}\csc\gamma\right]}f(y,z)\,dy\,dz$$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi r^2}}\sqrt{\frac{1-i\cot\theta}{2\pi s^2 i}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\left[\frac{\cot\beta}{2}\left(y^2+\left(u\frac{\csc\alpha}{r\csc\beta}\right)^2\right)-y\left(u\frac{\csc\alpha}{r\csc\beta}\right)\csc\beta-\frac{\cot\beta}{2}\left(u\frac{\csc\alpha}{r\csc\beta}\right)^2+\frac{\cot\beta}{2}u^2r^2\right]}$$
$$e^{\left[\frac{\cot\gamma}{2}\left(z^2+\left(v\frac{\csc\theta}{s\csc\gamma}\right)^2\right)-z\left(v\frac{\csc\theta}{s\csc\gamma}\right)\csc\gamma-\frac{\cot\gamma}{2}\left(v\frac{\csc\theta}{s\csc\gamma}\right)^2+\frac{\cot\gamma}{2}v^2s^2\right]}f(y,z)\,dy\,dz$$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi r^2}}\sqrt{\frac{1-i\cot\theta}{2\pi s^2 i}}e^{i\left[\frac{\cot\beta}{2}u^2r^2\left(1-\frac{\csc^2\alpha}{r^4\csc^2\beta}\right)\right]}e^{\left[\frac{\cot\gamma}{2}v^2s^2\left(1-\frac{\csc^2\theta}{s^4\csc^2\gamma}\right)\right]}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\left[\frac{\cot\beta}{2}\left(y^2+\left(u\frac{\csc\alpha}{r\csc\beta}\right)^2\right)-y\left(u\frac{\csc\alpha}{r\csc\beta}\right)\csc\beta\right]} e^{\left[\frac{\cot\gamma}{2}\left(z^2+\left(v\frac{\csc\theta}{s\csc\gamma}\right)^2\right)-z\left(v\frac{\csc\theta}{s\csc\gamma}\right)\csc\gamma\right]} f(y,z)\,dy\,dz$$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi r^2}}\sqrt{\frac{1-i\cot\theta}{2\pi s^2 i}}\, e^{i\left[\frac{\cot\beta}{2}u^2r^2\left(1-\frac{\csc^2\alpha}{r^4\csc^2\beta}\right)\right]} e^{\left[\frac{\cot\gamma}{2}v^2s^2\left(1-\frac{\csc^2\theta}{s^4\csc^2\gamma}\right)\right]}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\left[\frac{\cot\beta}{2}\left(x^2+\left(u\frac{\csc\alpha}{r\csc\beta}\right)^2\right)-x\left(u\frac{\csc\alpha}{r\csc\beta}\right)\csc\beta\right]} e^{\left[\frac{\cot\gamma}{2}\left(t^2+\left(v\frac{\csc\theta}{s\csc\gamma}\right)^2\right)-t\left(v\frac{\csc\theta}{s\csc\gamma}\right)\csc\gamma\right]} f(x,t)\,dx\,dt$$

$$=\sqrt{\frac{1-i\cot\alpha}{r^2(1-i\cot\beta)}}\sqrt{\frac{1-i\cot\theta}{s^2(1-i\cot\gamma)}}\, e^{i\left[\frac{\cot\alpha}{2}u^2\left(1-\frac{\csc^2\alpha}{r^4\csc^2\beta}\right)\right]} e^{\left[\frac{\cot\theta}{2}v^2\left(1-\frac{\csc^2\theta}{s^4\csc^2\gamma}\right)\right]}$$

$$\times\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u\frac{\csc\alpha}{r\csc\beta}, v\frac{\csc\theta}{s\csc\gamma}\right)$$

$$=\sqrt{\frac{1-i\cot\alpha}{r^2-i\cot\alpha}}\sqrt{\frac{1-i\cot\theta}{s^2-i\cot\theta}}\, e^{i\left[\frac{\cot\alpha}{2}u^2\left(1-\frac{\cos^2\alpha}{\cos^2\beta}\right)\right]} e^{\left[\frac{\cot\theta}{2}v^2\left(1-\frac{\cos^2\theta}{\cos^2\gamma}\right)\right]}\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u\frac{\sin\beta}{r\sin\alpha}, v\frac{\sin\gamma}{s\sin\theta}\right)$$

■

4.5 Differential properties

I $$\mathcal{FL}_{\alpha,\theta}\left\{\frac{\partial}{\partial x}f(x,t)\right\}(u,v) = -2ai\,\mathcal{FL}_{\alpha,\theta}\{xf(x,t)\} + iup\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\} \tag{4.7}$$

II $$\mathcal{FL}_{\alpha,\theta}\left\{\frac{\partial}{\partial t}f(x,t)\right\}(u,v) = -2b\,\mathcal{FL}_{\alpha,\theta}\{tf(x,t)\} + vq\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\} \tag{4.8}$$

Proof: By definition of FrFLT we have

(I) $$\mathcal{FL}_{\alpha,\theta}\left\{\frac{\partial}{\partial x}f(x,t)\right\}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}f(x,t)\,K_{\alpha,\theta}(x,u,t,v)\,dx\,dt$$

$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]}\frac{\partial}{\partial x}f(x,t)\,dx\,dt$$

$$= C\int_{-\infty}^{\infty} e^{[b(t^2+v^2)-qtv]}\left\{\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}\frac{\partial}{\partial x}f(x,t)\,dx\right\}dt$$

$$= C\int_{-\infty}^{\infty} e^{[b(t^2+v^2)-qtv]}\left\{\left[e^{i[a(x^2+u^2)-pxu]}f(x,t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}\,i[2ax-up]\,f(x,t)\,dx\right\}dt$$

$$= C\int_{-\infty}^{\infty} e^{[b(t^2+v^2)-qtv]}\left\{-\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}\,i[2ax-up]\,f(x,t)\,dx\right\}dt$$

$$= -2iaC\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}\,e^{[b(t^2+v^2)-qtv]}\,xf(x,t)dx\,dt$$

$$+iupC\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}\,e^{[b(t^2+v^2)-qtv]}\,f(x,t)\,dx\,dt$$

$$= -2ai\,\mathcal{FL}_{\alpha,\theta}\{xf(x,t)\} + iup\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}$$

(II) $$\mathcal{FL}_{\alpha,\theta}\left\{\frac{\partial}{\partial t}f(x,t)\right\}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}f(x,t)\,K_{\alpha,\theta}(x,u,t,v)\,dx\,dt$$

$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]}\frac{\partial}{\partial t}f(x,t)\,dx\,dt$$

$$= C\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}\left\{\int_{-\infty}^{\infty} e^{[b(t^2+v^2)-qtv]}\frac{\partial}{\partial t}f(x,t)\,dt\right\}dx$$

$$= C\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}\left\{\left[e^{[b(t^2+v^2)-qtv]}f(x,t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{[b(t^2+v^2)-qtv]}\,[2bt-vq]\,f(x,t)\,dt\right\}dx$$

$$= C\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}\left\{-\int_{-\infty}^{\infty} e^{[b(t^2+v^2)-qtv]}\,[2bt-vq]\,f(x,t)\,dt\right\}dx$$

$$= -2bC\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]}\, tf(x,t)\,dx\,dt$$

$$+ vqC\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]} f(x,t)\,dx\,dt$$

$$= -2b\,\mathcal{FL}_{\alpha,\theta}\{tf(x,t)\} + vq\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\} \qquad \blacksquare$$

4.6 Modulation properties

I $\mathcal{FL}_{\alpha,\theta}\{f(x,t)\cos(rx+ist)\}(u,v)$

$$= \frac{1}{2}e^{2ia\left(u-\frac{r}{2p}\right)\frac{r}{p}}e^{-2b\left(v+\frac{s}{2q}\right)\frac{s}{q}}\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u-\frac{r}{p},v+\frac{s}{q}\right) + \frac{1}{2}e^{-2ia\left(u+\frac{r}{2p}\right)\frac{r}{p}}e^{2b\left(v-\frac{s}{2q}\right)\frac{s}{q}}\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u+\frac{r}{p},v-\frac{s}{q}\right) \tag{4.9}$$

II $\mathcal{FL}_{\alpha,\theta}\{f(x,t)\sin(rx+ist)\}(u,v)$

$$= \frac{1}{2i}e^{2ia\left(u-\frac{r}{2p}\right)\frac{r}{p}}e^{-2b\left(v+\frac{s}{2q}\right)\frac{s}{q}}\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u-\frac{r}{p},v+\frac{s}{q}\right) - \frac{1}{2i}e^{-2ia\left(u+\frac{r}{2p}\right)\frac{r}{p}}e^{2b\left(v-\frac{s}{2q}\right)\frac{s}{q}}\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u+\frac{r}{p},v-\frac{s}{q}\right) \tag{4.10}$$

Proof: By definition of FrFLT we have,

(I) $\mathcal{FL}_{\alpha,\theta}\{f(x,t)\cos(rx+ist)\}(u,v)$

$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]} f(x,t)\cos(rx+st)\,dx\,dt$$

$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]} f(x,t)\left(\frac{e^{i(rx+ist)}+e^{-i(rx+ist)}}{2}\right)dx\,dt$$

$$= \frac{C}{2}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]} f(x,t)e^{i(rx+ist)}dx\,dt\right\} + \frac{C}{2}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]} f(x,t)e^{-i(rx+ist)}dx\,dt\right\}$$

$$= \frac{1}{2}\mathcal{FL}_{\alpha,\theta}\{f(x,t)e^{i(rx+ist)}\}(u,v) + \frac{1}{2}\mathcal{FL}_{\alpha,\theta}\{f(x,t)e^{-i(rx+ist)}\}(u,v)$$

$$= \frac{1}{2}e^{2ia\left(u-\frac{r}{2p}\right)\frac{r}{p}}e^{-2b\left(v+\frac{s}{2q}\right)\frac{s}{q}}\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u-\frac{r}{p},v+\frac{s}{q}\right) + \frac{1}{2}e^{-2ia\left(u+\frac{r}{2p}\right)\frac{r}{p}}e^{2b\left(v-\frac{s}{2q}\right)\frac{s}{q}}\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u+\frac{r}{p},v-\frac{s}{q}\right) \quad \text{By (4.2)}$$

(II) $\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\sin(rx+ist)\,(u,v)$

$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]} f(x,t)\sin(rx+ist)\,dx\,dt$$

$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]} f(x,t)\left(\frac{e^{i(rx+ist)}-e^{-i(rx+ist)}}{2i}\right)dx\,dt$$

$$= \frac{C}{2i}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]} f(x,t)e^{i(rx+ist)}dx\,dt\right\} - \frac{C}{2i}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i[a(x^2+u^2)-pxu]}e^{[b(t^2+v^2)-qtv]} f(x,t)e^{-i(rx+ist)}dx\,dt\right\}$$

$$= \frac{1}{2i}\mathcal{FL}_{\alpha,\theta}\{f(x,t)e^{i(rx+ist)}\}(u,v) - \frac{1}{2i}\mathcal{FL}_{\alpha,\theta}\{f(x,t)e^{-i(rx+ist)}\}(u,v)$$

$$= \frac{1}{2i}e^{2ia\left(u-\frac{r}{2p}\right)\frac{r}{p}}e^{-2b\left(v+\frac{s}{2q}\right)\frac{s}{q}}\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u-\frac{r}{p},v+\frac{s}{q}\right) - \frac{1}{2i}e^{-2ia\left(u+\frac{r}{2p}\right)\frac{r}{p}}e^{2b\left(v-\frac{s}{2q}\right)\frac{s}{q}}\,\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}\left(u+\frac{r}{p},v-\frac{s}{q}\right) \quad \text{By (4.2)} \qquad \blacksquare$$

Paper ID: SR25119122327 DOI: https://dx.doi.org/10.21275/SR25119122327

5. Conclusion

In this paper, we introduced the fractional Fourier-Laplace Transform (FrFLT), extending its applicability to all values of the fractional parameters α and θ. Previous studies on FrFLT primarily focused on specific parameter ranges, leaving certain cases undefined. To address this limitation, we extended a definition of the fractional Laplace Transform (FRLT) given by Torre. This extension completes the theoretical framework of the FrFLT, enabling a seamless operational calculus for FrFLT.

The main results of this paper include several derived and rigorously proved properties of the FrFLT, such as linearity, shifting, differentiation, scaling, and modulation. Furthermore, we analyzed the FrFLT under various parameter values, providing insights into its structure. These findings provide a theoretical framework for applying the FrFLT in diverse fields such as signal processing, engineering, and mathematical physics, offering new perspectives and expanding its range of applications. By establishing these foundational results, this work lays the groundwork for further advancements in the study and application of the FrFLT.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Funding

This research received no external funding.

References

[1] E. Sejdić, I. Djurović and L. Stanković, "Fractional Fourier transform as a signal processing tool: An overview of recent developments," *Signal Processing,* vol. 91, no. 6, pp. 1351-1369, 2011.

[2] H. M. Ozaktas, Z. Zalevsky and M. A. Kutay, The Fractional Fourier Transform: with Applications in Optics and Signal Processing, New York: Wiley, 2001.

[3] B. Barshan and B. Ayrulu, "Fractional Fourier transform pre-processing for neural networks and its applications to object recognition," *Neural Networks,* vol. 15, no. 1, pp. 131-140, 2002.

[4] A. W. Lohmann, Z. Zalevsky and D. Mendlovic, "Synthesis of pattern recognition filters for fractional Fourier processing," *Optics Communications* , vol. 128, no. 4-6, pp. 199-204, 1996.

[5] Q. Ran, H. Zhang, J. Zhang, L. Tang and J. Ma, "Deficiencies of the cryptography based on multiple-parameter fractional Fourier transform," *Optics Letters,* vol. 34, no. 11, pp. 1729-1731, 2009.

[6] T. Alieva, "Fractional Fourier transform as a tool for investigation of fractal objects," *Journal of the Optical Society of America A,* vol. 13, no. 6, pp. 1189-1192, 1996.

[7] N. Wiener, "Hermitian polynomials and Fourier analysis," *Journal of Mathematics and Physics,* vol. 8, no. 1-4, pp. 70-73, 1929.

[8] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *IMA Journal of Applied Mathematics,* vol. 25, no. 3, pp. 241-265, 1980.

[9] A. C. McBride and F. H. Kerr, "On Namias's Fractional Fourier Transforms," *IMA Journal of Applied Mathematics,* vol. 39, no. 2, pp. 159-175, 1987.

[10] A. Torre, "Linear and radial canonical transforms of fractional order," *Journal of Computational and Applied Mathematics,* vol. 153, no. 1, pp. 477-486, 2003.

[11] L. B. Almeida, "The Fractional Fourier Transform and Time-Frequency Representations," *IEEE Transactions on Signal Processing,* vol. 42, no. 11, pp. 3084-3091, 1994.

[12] K. K. Sharma, "Fractional Laplace transform," *Signal, Image and Video Processing*, vol. 4, pp. 377-379, 2010.

[13] H. M. Ozaktas and M. A. Kutay, "The fractional fourier transform," in *European Control Conference (ECC)*, Porto, Portugal, 2001.

[14] A. Torre, "Chapter 7- The fractional Fourier transform and some of its applications to optics," in *Progress in Optics*, vol. 43, E. Wolf, Ed., Elsevier, 2002, pp. 531-596.

[15] M. Omran and A. Kilicman, "Fractional double Laplace transform and its properties," in *2nd International Conference and Workshop on Mathematical Analysis* , 2016.

[16] P. R. Deshmukh and A. S. Gudadhe, "Fractional Quaternion Laplace Transform and Convolution," *International Journal of Contemporary Mathematical Sciences,* vol. 7, no. 18, pp. 857-863, 2012.

[17] V. D. Sharma, "Operational Calculus on Generalized Two-Dimensional Fractional Fourier Transform," *International Journal of Engineering and Innovative Technology (IJEIT),* vol. 3, no. 2, pp. 253-256, 2013.

[18] V. D. Sharma and A. N. Rangari, "Operational Calculus on Generalized Fourier-Laplace Transform," *International Journal of Scientific and Innovative Mathematical Research ,* vol. 2, no. 11, pp. 862-867, 2014.

[19] V. Sharma and A. Patalwanshi, "Convolution Structure of the Fractional Fourier-Laplace Transform," *International Journal of Science and Research (IJSR),* vol. 13, no. 8, pp. 1599-1602, 2024.