# A Study of Fixed Point for Two Continuous Random Operators in 2-Hilbert Space

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**Abstract:** ***In this paper, we construct a sequence of measurable functions and examine its convergence to the common random fixed point of two continuous random operators defined on a non-empty closed subset of a separable 2-Hilbert space. To obtain the random fixed point of these operators, we employ a rational inequality and the parallelogram law.***

**2020 Mathematical Sciences Classification:** 54H25, 47H10.

**Keywords:** Separable 2-Hilbert Space, Random operators, Measurable function, Common random fixed point.

## 1. Introduction

In recent years, the investigation of random fixed points has gained significant interest, with several recent studies in this area being referenced in [7, 14]. In this work, we construct a sequence of measurable functions and analyze its convergence towards the unique random fixed point shared by two continuous random operators, which are defined on a non-empty closed subset of a separable Hilbert space. To obtain the random fixed point for these two operators, we employ a rational inequality and utilize the parallelogram law. In this paper, $(\Omega, \Sigma)$ represents a measurable space consisting of the set $\Omega$ and the sigma-algebra $\Sigma$ of subsets of $\Omega$. The symbol H denotes a separable 2-Hilbert space, while C refers to a non-empty closed subset of H.

## 2. Preliminaries

**Definition 2.1.** A function $f: \Omega \to C$ is said to be measurable if $f^{-1}(B \cap C) \in \Sigma$ for every Borel subset $B$ of $H$. **Definition 2.2.** A function $F: \Omega\times C \to C$ is said to be a random operator if $F(., x): \Omega \to C$ is measurable for every $x \in C$.

**Definition 2.3.** A measurable function $g: \Omega \to C$ is said to be a random fixed point of the random operator $F: \Omega \times C \to C$ if $F(t, g(t)) = g(t)$ for all $t \in \Omega$.

**Definition 2.4.** A random operator $F: \Omega\times C \to C$ is said to be continuous if for fixed $t \in \Omega$, $F(t,.): C \to C$ is continuous.

**Condition (A).** Two mappings $P, Q: C \to C$, where $C$ is a non-empty closed subset of a Hilbert space $H$, is said to satisfy condition (A) if
$\|Px - Qy, t\|^2 \le a_1\|x - y, t\|^2 + a_2\left[\|x - Px, t\|^2 + \|y - Qy, t\|^2\right] + a_3 \frac{\|y-Qy,t\|^2[1+\|x-Px,t\|^2]}{1+\|x-y,t\|^2}$
for each x, y $\in$ C, $a_1, a_2$ being positive real numbers such that $0 < a_1 + a_2 + a_3 < \frac{1}{2}$

## 3. Main Result

**Theorem 3.1:** Let C be a non-empty closed subset of a separable 2-Hilbert space H. Let P and Q be two continuous random operators defined on C such that for $t \epsilon \Omega$, P $(t,.)$, Q $(t,.)$: $C \to C$ satisfy condition (A). Then P and Q have a common unique random fixed point in C.

**Proof:** We define a sequence of functions $\{g_n\}$ $as$ $g_0 \in$ C is arbitrary measurable function for t $\in \Omega$ and n = 0, 1, 2, 3. . . . . . .
$g_{2n+1}(t)$ = P (t, $g_{2n}(t)$), $g_{2n+2}(t)$ = Q (t, $g_{2n+1}(t)$) … (3.1.1)

If $g_{2n}(t) = g_{2n+1}(t) = g_{2n+2}(t)$ for t $\in \Omega$ for some n then we see that $g_{2n}(t)$ a random fixed point of P and Q. So, we assume that no two consecutive terms of sequence $\{gn\}$ are equal.
For $t \epsilon \Omega$,
$\|g_{2n+1}(t) - g_{2n+2}(t), a\|^2 = \|P(t, g_{2n}(t)) - Q(t, g_{2n+1}(t)), a\|^2$
$\le a_1\|g_{2n}(t) - g_{2n+1}(t), a\|^2$
$+a_2[\|g_{2n}(t) - P(t, g_{2n}(t)), a\|^2 + \|g_{2n+1}(t) - Q(t, g_{2n+1}(t)), a\|^2]$
$+ a_3 \frac{\|g_{2n+1}(t) - Q(t, g_{2n+1}(t)), a\|^2[1+\|g_{2n}(t) - P(t, g_{2n}(t)), a\|^2]}{1+\|g_{2n}(t)-g_{2n+1}(t), a\|^2}$
$= a_1\|g_{2n}(t) - g_{2n+1}(t), a\|^2$
$+ a_2[\|g_{2n}(t) - g_{2n+1}(t)), a\|^2 + \|g_{2n+1}(t) - g_{2n+2}(t), a\|^2]$
$+ a_3 \frac{\|g_{2n+1}(t) - g_{2n+2}(t)\, a\|^2[1+\|g_{2n}(t) - g_{2n+1}(t), a\|^2]}{1+\|g_{2n}(t)-g_{2n+1}(t), a\|^2}$
$= (a_1 + a_2)\|g_{2n}(t) - g_{2n+1}(t), a\|^2 + (a_2 + a_3)\|g_{2n+1}(t) - g_{2n+2}(t)\, a\|^2$
$\Rightarrow (1 - a_2 + a_3)\|g_{2n+1}(t) - g_{2n+2}(t)\, a\|^2 \le (a_1 + a_2)\|g_{2n}(t) - g_{2n+1}(t), a\|^2$
$\Rightarrow \|g_{2n+1}(t) - g_{2n+2}(t)\, a\|^2 \le K\|g_{2n}(t) - g_{2n+1}(t), a\|^2$
Where, K = $\frac{(a_1+a_2)}{(1-a_2-a_3)} < 1$ … (3.1.2)

In general,
$\|g_n(t) - g_{n+1}(t)\, a\|^2 \le K\|g_{n-1}(t) - g_n(t), a\|^2$
$\Rightarrow \|g_n(t) - g_{n+1}(t)\, a\| \le K^n\|g_{n-1}(t) - g_n(t), a\|$ for t $\in \Omega$

Now, we shall prove that for t $\in \Omega$, $\{g_n(t)\})$ } is a Cauchy sequence.
For every position integer n, we have
$\|g_n(t) - g_{n+p}(t)\, a\| = \|g_n(t) - g_{n+1}(t) - g_{n+2}(t) - \ldots\ldots - g_{n+p-1}(t) - g_{n+p}(t), a\|$

For all $t \epsilon \Omega$, $\{gn\ (t)\ \}$ is a Cauchy sequence. for this for every position integer i we have, for $t \epsilon \Omega$.
$\|\ gn\ (t) - gn{+}p\ (t),\ a\ \| = \|\ gn\ (t) - gn{+}1\ (t) + \cdots + gn{+}p{-}1\ (t) + -gn{+}p\ (t),\ a\ \|$

$\leq \|\ gn\ (t) - gn{+}1\ (t),\ a\ \| + \|\ gn{+}1\ (t) - gn{+}2\ (t),\ a\ \| + \ldots + \|\ gn{+}p{-}1\ (t) - gn{+}p\ (t),\ a\ \|$
$\leq [K^n + K^{n+1} + \ldots + K^{n+p-1}]\ \|g0\ (t) - g1\ (t), a\|$
$= K^n\ [1 + K + K^2 + \ldots + K^{p-1}]\ \|g0\ (t) - g1\ (t), a\|$
$= \frac{K^n}{1-K}\ \|g0\ (t) - g1\ (t), a\|$, for all $t \epsilon \Omega$
As n → ∞, $\|gn\ (t) - gn{+}p\ (t), a\| \to 0$. It follows that for all $t \epsilon \Omega$, $\{gn\ (t)\ \}$ is a Cauchy sequence and hence is convergent in 2-Hilbert Space H.

**Existence of random fixed point:** for all $t \epsilon \Omega$,
Let $gn\ (t) \to$ g (t) as n → ∞ … (3.1.3)
Since C is closed and g is a function from C to C.

By Parallelogram law, we have
$\|g(t) - Q\ (t, g(t)), a\|^2 = \|g(t) - g_{2n+1}(t) + g_{2n+1}(t) - Q\ (t, g(t)), a\|^2$
$\leq 2\|g(t) - g_{2n+1}(t), a\|^2 + 2\|g_{2n+1}(t) - Q\ (t, g(t)), a\|^2$
$= 2\|g(t) - g_{2n+1}(t), a\|^2 + 2\|\mathrm{P}\ (t, g_{2n}\ (t)) - Q\ (t, g(t)), a\|^2$
$\leq 2\{\|g(t) - g_{2n+1}(t), a\|^2 + a_1\|g_{2n}\ (t) - g(t), a\|^2 + a_2[\|\ g_{2n}\ (t) - \mathrm{P}\ (t,\ g_{2n}\ (t)), a\|^2 + \|g\ (t) - Q\ (t, g(t)), a\|^2\ ]$
$+ a_3 \frac{\|g(t)-Q\ (t,g(t)),a\|^2\left[1+\ \|g_{2n}(t)-\mathrm{P}(t,\ g_{2n}(t)),a\|^2\right]}{1+\ \|g_{2n}(t)-g(t),a\|^2}\}$

As $\{g2n\ (t)\ \}$ and $\{g2n{+}1\ (t)\ \}$ are sub sequences of $\{gn\ (t)\ \}$, hence as n → ∞, $\{g2n\ (t)\ \} \to$ g (t) and $\{g2n{+}1\ (t)\ \} \to$ g (t).

Therefore, as n → ∞,
$\|g(t) - Q\ (t, g(t)), a\|^2 \leq 2\{\|g(t) - g(t), a\|^2 + a_1\|g(t) - g(t), a\|^2$
$+ a_2[\|\ g\ (t) - g\ (t), a\|^2 + \|g\ (t) - Q\ (t, g(t)), a\|^2\ ]$
$+ a_3 \frac{\|g(t)-Q\ (t,g(t)),a\|^2[1+\ \|g(t)-g\ (t),a\|^2]}{1+\ \|g(t)-g(t),a\|^2}$
$\Rightarrow \|g(t) - Q\ (t, g(t)), a\|^2 \leq 2\ (a_2 + a_3)\ \|g\ (t) - Q\ (t, g(t)), a\|^2$
$\Rightarrow (1\text{-}2a_2\text{-}2a_3)\ \|g(t) - Q\ (t, g(t)), a\|^2 \leq 0$
$\Rightarrow (1\text{-}2a_2\text{-}2a_3)\ \|g(t) - Q\ (t, g(t)), a\|^2 = 0$ (as $2\ (a_2 + a_3) < 1$)

This implies that
Q (t, g (t)) = g (t) for all $t \epsilon \Omega$,
Similarly, P (t, g (t)) = g (t) for all $t \epsilon \Omega$

Again, If S: Ω × C → C be a continuous random operator on a non-empty subset C of a Separable 2-Hilbert space H, then for any measurable function f: Ω → C, the function g (t) = S (t, f (t) h (t) is also measurable [9].

It follows from the construction of $\{gn\ (t)\ \}$ $(by\ (3.1.1))$ and the above consideration that $\{gn\ (t)\ \}$ is a sequence of measurable function. This fact shows that g: $\Omega \to C$ is a common random fixed point of P and Q. This completes the proof.

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