Cordial Labeling of $K_{n,n}$ related graphs

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***Abstract*—It is proved by cahit that $K_{m,n}$ is cordial for all m and n. In this paper cordial labeling for three graphs related to complete bipartite graph $K_{n,n}$ is discussed. We prove that (1) star of $K_{n,n}$, (2) path union of $K_{n,n}$, and (3) the graph obtained by joining two copies of $K_{n,n}$ by a path of arbitrary length are cordial graphs.**
Key words : complete bipartite graph $K_{n,n}$, Cordial graph, Path union.
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I. Introduction

We begin with simple, finite, undirected graph $G = (V, E)$. In this paper P_n denotes path with n vertices and $K_{n,n}$ denotes complete bipartite graph with $2n$ vertices. For all other terminology and notations we follow Harary[4].

Definition 1.1 A graph $G = (V, E)$ is said to be *bipartite graph* if the vertex set can be partitioned into two subsets V_1 and V_2 such that for every edge $e_i = v_i v_j \epsilon E$, $v_i \epsilon V_1$ & $v_j \epsilon V_2$.

Definition 1.2 A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. If partite sets are having m and n vertices then the related complete bipartite graph is denoted by $K_{m,n}$.

Definition 1.3 A graph obtained by replacing each vertex of star graph $K_{1,n}$ by a graph G is called *star of G*. We denote it as G^*.

Definition 1.4 Let G be a graph and $G_1, G_2, \ldots, G_n$, $n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} (for $i = 1, 2, \ldots, n-1$) is called *path union of G*.

Definition 1.5 If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

A dynamic survey of graph labeling is published and updated every year Gallian[3]. The reference cited here is of the updated survey of 2012.

Definition 1.6 Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called *label of the vertex v* of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ denote the number of vertices of G with labels 0, 1 respectively under f and let $e_f(0)$, $e_f(1)$ denote the number of edges of G with labels 0, 1 respectively under f^*.

Definition 1.7 A binary vertex labeling of a graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *cordial* if it admits cordial labeling.

II. Literature survey and Previous work

The concept of cordial graphs was introduced by Cahit[1]. Cahit[2] proved that complete bipartite graphs $K_{m,n}$ are cordial for all m and n. Shee and Ho[6] proved that path union of cycles, petersen graphs, trees, wheels, unicyclic graphs are cordial. Vaidya et al.[7] proved that the graph obtained by joining two copies of cycles by a path of arbitrary length is cordial. In [8], the same authors proved that path union of cycle with one chord is cordial. Vaidya et al. [9] proved that Star of cycle C_n^* is cordial for all n. In [10], same authors proved that star of petersen graph, the graph obtained by joining two copies of petersen graph by a path of arbitrary length and the graph obtained by joining two copies of wheel graph by a path of arbitrary length are cordial graphs. Selvaraju[5] proves that the one-point union of any number of copies of a complete bipartite graph is cordial.

III. Main Results

Theorem 3.1 Star of complete bipartite graph $K_{n,n}$ is cordial.

Proof: Let $G = (K_{n,n})^*$ be the star of complete bipartite graph $K_{n,n}$. Let V_1, and V_2 be the partitions of the vertex set of the central copy of complete bipartite graph $K_{n,n}$ in G. Let $v_1, v_2, ..., v_n$ be successive vertices of the set V_1 and $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ be successive vertices of the set V_2 (in counter clockwise direction). Let V_{i1} and V_{i2} be the partitions of the vertex set of i^{th} copy of complete bipartite graph $K_{n,n}$ in G (except central one). Let $u_{i1}, u_{i2}, \ldots, u_{in}$ be successive vertices of the set V_{i1} and let $u_{i(n+1)}, u_{i(n+2)}, \ldots, u_{i(2n)}$ be successive vertices of the set V_{i2}.Let $e_i = u_{i1}v_i$ be the edge joining central copy and i^{th} copy of $K_{n,n}$. Moreover let $u_{ij}^{(0)}$ denote the vertices of the copy of $K_{n,n}$ which is joined to vertex with label 0 of the central copy of $K_{n,n}$ by an edge and let $u_{ij}^{(1)}$ denote the vertices of the copy of $K_{n,n}$ which is joined to vertex with label 1 of the central copy of $K_{n,n}$, where $i = 1, 2, \ldots, 2n$, $j = 1, 2, \ldots, 2n$. To define required labeling $f : V(G) \rightarrow 0, 1$ we consider the following cases.

Case 1: $n \equiv 1, 3(mod4)$
For $1 \leq i \leq 2n$,
$f(v_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$
$f(u_{ij})^{(0)} = 0$; if $j \equiv 0, 1(mod4)$
$= 1$; if $j \equiv 2, 3(mod4)$, $1 \leq j \leq 2n$
$f(u_{ij})^{(1)} = 0$; if $j \equiv 2, 3(mod4)$
$= 1$; if $j \equiv 0, 1(mod4)$, $1 \leq j \leq 2n$

Case 2: $n \equiv 1, 3(mod4)$
$f(v_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$i \equiv 1, 2(mod4)$
$f(u_{ij}) = 0$; if $j \equiv 0, 3(mod4)$
$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq j \leq 2n$
$i \equiv 0, 3(mod4)$
$f(u_{ij}) = 0$; if $j \equiv 1, 2(mod4)$
$= 1$; if $j \equiv 0, 3(mod4)$, $1 \leq j \leq 2n$

The graph G under consideration satisfies the condition $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ in each case which is shown in *Table*. Hence the graph G is cordial graph.

Let $n = 4a + b, k = 4c + d$, where $n, k \in N$.

TABLE I
TABLE FOR STAR OF COMPLETE BIPARTITE GRAPH $K_{n,n}$

b	vertex conditions	edge conditions
0,1,2,3	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$

Illustration 3.1 The cordial labeling of star of complete bipartite graph $K_{3,3}$ is shown in *Figure 1* as an illustration for the proof of *Theorem 3.1*. It is the case related to $n \equiv 3(mod4)$.

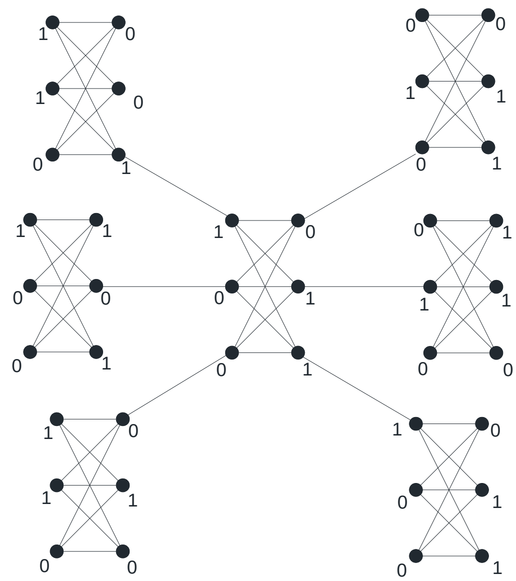

Fig. 1. Cordial labeling of star of complete bipartite graph $K_{3,3}$

Theorem 3.2 The path union of k copies of complete bipartite graph $K_{n,n}$ is cordial for all $n \geq 2$.

Proof: Let G be path union of k copies of complete bipartite graph $K_{n,n}$ and let $G_1, G_2, \ldots, G_k$ be k copies of complete bipartite graph $K_{n,n}$. Let $u_{i1}, u_{i2}, \ldots, u_{i2n}$, for $i = 1, 2, \ldots, k$ denote the successive vertices (in counter clockwise direction) of graph G_i. Let $e_i = u_{i1}u_{(i+1)n}$ be the edge joining G_i and G_{i+1}, for $i = 1, 2, \ldots, k-1$. Here we define labeling function $f : V(G) \rightarrow \{0, 1\}$ as follows.

Case 1: $n \equiv 1, 3(mod4)$
$i \equiv 1, 3(mod4)$
$f(u_{ij}) = 0$; if $j \equiv 2, 3(mod4)$
$= 1$; if $j \equiv 0, 1(mod4)$, $1 \leq j \leq 2n$
$i \equiv 0, 2(mod4)$
$f(u_{ij}) = 0$; if $j \equiv 0, 1(mod4)$
$= 1$; if $j \equiv 2, 3(mod4)$, $1 \leq j \leq 2n$

Case 2: $n \equiv 0(mod4)$
$i \equiv 0, 2(mod4)$
$f(u_{ij}) = 0$; if $j \equiv 0, 3(mod4)$
$= 1$; if $j \equiv 1, 2(mod4)$, $1 \leq j \leq 2n$
$i \equiv 1, 3(mod4)$
$f(u_{ij}) = 0$; if $j \equiv 1, 2(mod4)$
$= 1$; if $j \equiv 0, 3(mod4)$, $1 \leq j \leq 2n$

Case 3: $n \equiv 2(mod4)$
$i \equiv 1, 2(mod4)$
$f(u_{ij}) = 0$; if $j \equiv 2, 3(mod4)$
$= 1$; if $j \equiv 0, 1(mod4)$, $1 \leq j \leq 2n$
$i \equiv 0, 3(mod4)$
$f(u_{ij}) = 0$; if $j \equiv 0, 1(mod4)$
$= 1$; if $j \equiv 2, 3(mod4)$, $1 \leq j \leq 2n$

Illustration 3.2 The cordial labeling of path union of four copies of complete bipartite graph $K_{4,4}$ is

TABLE II
TABLE FOR PATH UNION OF COMPLETE BIPARTITE GRAPH $K_{n,n}$

b	d	vertex conditions	edge conditions
0	1,3	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
	0,2	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
1,3	0,1,2,3	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
2	1,3	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
	0,2	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
	2	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$

shown in *Figure 2* as an illustration for the proof of *Theorem 3.2*. It is the case related to $n \equiv 0(mod4)$, $k \equiv 0(mod4)$.

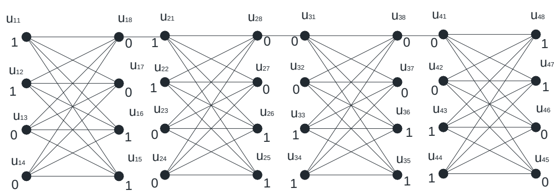

Fig. 2. Cordial labeling of path union of 4 copies of $K_{4,4}$

Theorem 3.3 The graph obtained by joining two copies of complete bipartite graph $K_{n,n}$ by a path of arbitrary length is cordial.

Proof: Let G be the graph obtained by joining two copies of complete bipartite graph $K_{n,n}$ by a path of arbitrary length. Let us denote the successive vertices (in counter clockwise direction) of first copy of complete bipartite graph by $u_1, u_2, \ldots, u_{2n}$ and the successive vertices (in counter clockwise spiral direction) of second copy of complete bipartite graph by $w_1, w_2, \ldots, w_{2n}$. Let $v_1, v_2, \ldots, v_k$ be the vertices of path P_k with $v_1 = u_1$ and $v_k = w_n$. Here we define labeling function $f : V(G) \to \{0, 1\}$ as follows.

Case 1: $n \equiv 2(mod4)$

Subcase I: $k \equiv 1(mod4)$

$f(u_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(w_i) = 0$; if $i \equiv 0, 1(mod4)$
$= 1$; if $i \equiv 2, 3(mod4)$, $1 \leq i \leq 2n$
$f(v_j) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq j \leq k$

Subcase II: $k \equiv 0, 2(mod4)$

$f(u_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(w_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(v_j) = 0$; if $i \equiv 0, 1(mod4)$
$= 1$; if $i \equiv 2, 3(mod4)$, $1 \leq j \leq k$

Subcase III: $k \equiv 3(mod4)$

$f(u_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(w_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(v_j) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq j \leq k$

Case 2: $n \equiv 0(mod4)$

Subcase I: $k \equiv 1(mod4)$

$f(u_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(w_i) = 0$; if $i \equiv 0, 1(mod4)$
$= 1$; if $i \equiv 2, 3(mod4)$, $1 \leq i \leq 2n$
$f(v_j) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq j \leq k$

Subcase II: $k \equiv 2, 3, 4(mod4)$

$f(u_i) = 0$; if $i \equiv 2, 3(mod4)$,
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(w_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(v_j) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq j \leq k$

Case 3 $n \equiv 3, 5(mod4)$

Subcase I $k \equiv 2, 3(mod4)$

$f(u_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(w_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(v_j) = 0$; if $i \equiv 0, 1(mod4)$
$= 1$; if $i \equiv 2, 3(mod4)$, $1 \leq j \leq k$

Subcase II $k \equiv 1(mod4)$

$f(u_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(w_i) = 0$; if $i \equiv 0, 1(mod4)$
$= 1$; if $i \equiv 2, 3(mod4)$, $1 \leq i \leq 2n$
$f(v_j) = 0$; if $i \equiv 1, 2(mod4)$
$= 1$; if $i \equiv 0, 3(mod4)$, $1 \leq j \leq k$

Subcase III $k \equiv 0(mod4)$

$f(u_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(w_i) = 0$; if $i \equiv 2, 3(mod4)$
$= 1$; if $i \equiv 0, 1(mod4)$, $1 \leq i \leq 2n$
$f(v_j) = 0$; if $i \equiv 1, 2(mod4)$
$= 1$; if $i \equiv 0, 3(mod4)$, $1 \leq j \leq k$

Illustration 3.3 The cordial labeling of the graph G obtained by joining two copies of complete bipartite graph $K_{5,5}$ by path P_8 is shown in *Figure 3* as an illustration for the proof of *Theorem 3.3*. It is the case related to $n \equiv 1(mod4)$ and $k \equiv 2(mod4)$.

TABLE III
TABLE FOR THE GRAPH G IN THEOREM 3.3

b	d	vertex conditions	edge conditions
0	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
	1	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1)$
	2	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
	3	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
1	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
	1	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	2	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
	3	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1)$
2	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
	1	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	2	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
	3	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
3	0	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 1$
	1	$v_f(0) = v_f(1) + 1$	$e_f(0) = e_f(1)$
	2	$v_f(0) = v_f(1)$	$e_f(0) + 1 = e_f(1)$
	3	$v_f(0) + 1 = v_f(1)$	$e_f(0) = e_f(1)$

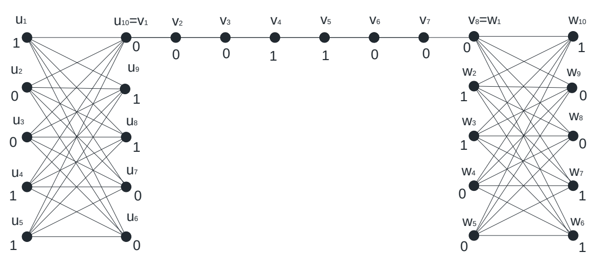

Fig. 3. Cordial labeling of graph obtained by joining two copies of $K_{5,5}$ by path P_8

IV. CONCLUSION

We relate the cordial labeling of a graph with the operations star of a graph and path union of a graph which contributes two new results in the theory of cordial graphs. The third result is based on cordiality of a graph obtained by two copies of a graph by a path of arbitrary length. Thus through research work in this paper, we found three new cordial graphs. We have provided tables and sufficient illustrations for the better understanding of the techniques used in the proof of the theorems.

V. SCOPE OF FURTHER RESEARCH

We have proved cordiality of star of $K_{n,n}$ and path union of $K_{n,n}$, for all n. These results may be extended to the star of $K_{m,n}$ and path union of $K_{m,n}$, for all m and n. One may think about similar results for complete graph K_n for all n.

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