Generalised Topological Spaces: A Fundamental Exploration

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Abstract: ***The concept of topological space is often generalised by replacing open sets with other kind of subsets. In many cases, a generalised topology is a subset. μ of P(X) that contains ϕ and any union of elements of μ belongs to μ. In this article, we review the concept of generalised topological spaces.***

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1. Introduction

A topological space typically comprises a set accompanied by a specific topological structure that defines its open sets. This well-established framework allows for the demonstration of various properties and theorems within Mathematics. However, an intriguing question arises when considering the implications of replacing traditional topology with a more generalised structure. The exploration of this concept began in 1963 when mathematician Lev Levin introduced the ideas of semi-open sets and semi-continuity within topological spaces. This innovative approach sparked interest among researchers, driving them to investigate weaker forms of open sets and ultimately leading to the development of a broader, more generalised framework for understanding topological properties. The ongoing research in this area continues to unveil new insights into the nature of continuity and convergence, thus expanding the horizons of topological theory. This paper is an attempt to review topological spaces.

For example, in 1965, O. N. Jasted introduced the notation of α-open space; in 1982, A. S. Mashhour introduced pre-open sets, and R. A. Mahmoud introduced β-open sets. Finally, in 1997, A. Csaszar generalised these new open sets by introducing the concept of ϒ-open sets. He defined the concept of generalised topology in 2002. A subset μ of the power set of a nonempty set X is called generalised topology on X if μ has the following properties

1) ϕ ∈μ.
2) Any union of elements of μ belongs to μ.

Generalised topological space is an important generalisation of topological spaces that helps to prove many interesting results. This paper aims to introduce the concept of generalized topological spaces. In generalised topology, the role of an open set is given to μ-open sets, where μ is a generalised topology, so this concept belongs rather to the theory of generalized topological spaces instead of topology in the strict sense.

Generalised Topological Spaces

The concept of topological space is often generalised by replacing open sets with other kind of subsets. In many cases, a generalised topology is a subset. μ of P(X) that contains ϕ and any union of elements of μ belongs to μ.

Definition Let X be any set, a subset μ of P(X) is called generalised topology on X if μ has the following properties

1) $\phi \in \mu$
2) Any union of elements of μ belongs to μ.

The pair (X,μ) is called generalised topological space.

Remark: In generalised topology, we replace the family of open sets with larger ones. In general, every topology is a generalised topology. The union of all elements of μ will be denoted by Mμ.

Example

Let X={1, 2, 3} and $\mu = \{\phi, X, \{1,2\}, \{1,3\}, \{2,3\}\}$ are generalised topology on X.

Definition Let $B \subset P(X)$ and $\phi \in B$, then B is called base for μ if {{∪ B': B' $\subset B$}=μ.we also say that μ is generated by B.

Definition: A generalised topology is called strong if $X \in \mu$.

Example

Indiscrete topology is a strong topology since the only elements are ϕ and X.

Definition A generalised topological space is called a quasi-topological space if μ is closed under finite intersections**.**

Example

Let X=ℝ and μ={ϕ, {1}, {1,2}, {1,2,3}}

Definition A subset B of X is called μ-open if B∈ μ.

Definition A subset B of X is called. μ-closed if $X - B \in \mu$.

Example

Let $X = \boldsymbol{I_n} = \{1,2,3,\ldots..n\}$ define $\mu = \{\phi, X\}, \cup\{A \subseteq \mathrm{I_n}: \mathrm{A} = \mathrm{I_n} - \{\mathrm{i}\},\ \mathrm{i} = 1,2,3 \ldots.\}$ then μ is a generalised topology and μ also strong. Here the only μ-closed subsets are ϕ and X and singleton subsets of $\mathbf{I_n}$. Then μ is called the generalised topology and (X, μ) is called co-singleton generalised topological space.

Definition For $B \subset X$ let I(B)is the largest μ-open subset of B or I(B) is the union of all μan open subset of B, then I(B) is called the interior of B.

Definition Let C(B) be the smallest μ- closed subset, which contains B and C(B), is called the closure of B.

Definition: A point x∈X is called a μ-cluster point of B if $U \cap (\mathrm{B} - \{\mathrm{X}\}) \neq \phi$ for each $U \in \mu$ with $x \in U$.the set of all μ-cluster points of B is denoted by d(B). The collection of all μ-open sets that contains a point x is denoted by μ_x. that is $\mu_x = \{\mathrm{U}: \mathrm{U} \in \mu, x \in U\}$.

Proposition

Let B be a subset of a space X. Then the following holds

1) $I(B) \subseteq B \subseteq C(B)$
2) $I(I(B)) = I(B)$ and $C(C(B)) = C(B)$
3) $If\ B' \subseteq B\ then\ I(B') \subseteq I(B)$ and $C(B') \subseteq C(B)$ and $d(B') \subseteq d(B)$
4) $I(B) = B \Leftrightarrow B$ is μ −open
5) $C(B) = B \Leftrightarrow B$ is μ-closed
6) $C(B) = X - I(X - B)$ and $I(B) = X - C(X - B)$
7) $C(B) = B \cup d(B)$
8) $x \in C(B) \Leftrightarrow U \cap B \neq \phi$ for each $U \in \mu_x$
9) $x \in I(B) \Leftrightarrow U \subseteq B$ for some $U \in \mu_x$.
10) $x \neq d(\{x\})$ for each $x \in X$

Proof

1) Let (X,μ) be a generalized topological space and B⊂X. From the definition we have

$$\mathrm{I(B)} = \cup\{\mathrm{G} \in \mu: \mathrm{G} \subset \mathrm{B}\}$$
$$I(B) \subseteq B \qquad (1)$$

And C(B)=∩{F:F is μ-closed and F⊇ B}

$$\mathrm{B} \subseteq \mathrm{C(B)} \qquad (2)$$

From (1) and (2) we get I(B) ⊆B⊆C(B).

2) I(I(B) is the largest μ-open set contained in B.I(I(B)) is the largest μ-opren set contained in I(B) is open I(I(B))is the largest μ-open set contained in B.
hence $I(I(B)) = I(B)$.
C(B) is the smallest μ-closed set contained in B.C(C(B)) is the smallest μ-closed set contains C(B).since C(B) is closed, C(C(B)) is the C(B) is the smallest μ-closed set contains B.
Hence $C\big(C(B)\big) = C(B)$.

3) If $B' \subseteq B$ we If B have $I(B') \subseteq B'$
So if, $B' \subseteq B$ then $I(B') \subseteq B$
Thus I(B′)is open set contained in B.
so $I(B') \subseteq I(B)$
Since $B \subseteq C(B)$,if B′ is contained in B, we have $B' \subseteq C(B)$
Since C(B) is closed, we must have $C(B') \subseteq C(B)$.

4) Assuming $I(B) = B$
I(B) is the largest μ-open set contained in B, so B is μ-open.

Conversely assume B is μ-open, that is B is san open set contained in B.
$B \subseteq I(B)$ (1)

(Since I(B) is the union of all μ-open set contained in B) also we have
$I(B) \subseteq B$. (2)
From (1) and (2) $I(B) = Bi_\mu$.

5) Assume $C(B) = B$.
Since C(B) is the smallest μ-closed set contains B, B is μ-closed.
Conversely assume B is μ-closed.
i.e,B is closed set containing B. So
$C(B) \subseteq B$. (1)
(Since C(B) is the intersection of all μ-closed set) also we have
$B \subseteq C(B)$ (2)
From (1) and (2) $C(B) = B$.

6) To prove $[C(B)]^{\mathrm{C}} = I(X - B)$, we have

$$C(B) = \cap\{F: F \text{ is closed and } B \subseteq F\}$$
$$[C(B)]^{\mathrm{C}} = \cup\{Fc: F \text{ is closed and } B \subseteq F\}$$
$$= \cup\{U: U \text{ is open and } B \subseteq U^{\mathrm{C}}\}$$
$$= \cup\{U: U \text{ is open and } U \subseteq B^{\mathrm{C}}\} = I(X - B)$$
$$\text{Hence } C(B) = X - I(X - B).$$

To prove $[I(B)]^{\mathrm{C}} = C(X - B)$,we have

$$I(B) = \cup\{U: U \text{ is open and } U \subseteq B\}$$
$$[I(B)]^c = \cap\ \{U^c: U \text{ is open and } U \subseteq B\}$$
$$= \cap\ \{F: F \text{ is closed and } F^c \subseteq B\} = C(X - B) = \{F: F\ is\ closed\ and\ B^c \subseteq F\}$$
$$Hence\ I(B) = X - C(X - B).$$

7) Claim $B \cup d(B)$ is closed or $X - B \cup d(B)$ is open.
To show that $B \cup$ d(B) a neighbour of each of its points.
Let $y \in B \cup d(B)$ then since y is not a point of accumulation of B, no point of B except y. But $y \notin B$.so $B \cap V = \phi$
We claim $d(B) \cap V = \phi$ V is an open set containing z which is an accumulation point of B.so $V \cap B = \phi$.this is a contradiction.so $d(B) \cap V = \phi$ and $V \subset X - B \cup d(B)$.this proves that $(B\ B \cup d)$ is closed, and since it contains B. It also contains C(B).
i.e, $C(B) \subset B \cup d(B)$ (1)
Other way inclusion $B \cup d(B)$~~$\in C(B)$, it suffices to~~ show that $d(B) \subset C(B)$.
Since we have $B \subset C(B)$ so let $y \in d(B)$
If $y \notin C(B)\ then\ y \in X - C(B)$ which is open set since C(B) is always a closed set but y is an accumulation point of B.
So $(X - C(B)) \cap B \neq \phi$ which is a contradiction.
Since $X - C(B) \subset X - B.so\ y \in C(B)$.
Hence $B \cup d(B) \subset C(B)$. (2)
From (1) and (2) , $C(B) = B \cup d(B)$.

8) Let $D = \{x \in X: U \cap B \neq \phi, U \in \mu_x\}$.
Let $x \in C(B)$. so $x \in B \cup d(B)$. then $x \in B$ or $x \in d(B)$.
Let $x \in U\ and\ U \in \mu_x$.
If $x \in B, U \cap B \neq \phi$.

If $x \in d(B)$ then $\forall$ open set containing x contains a point of B other than x.since U is neighborhood of x,$\exists$ open set V such that $x \in V$ and $V \subset U$.
Here V is an open set x∈V .so $V \cap B \neq \phi$(since $x \in d(B)$).
Hence $U \cap B \neq \phi$ *for each* $U \in \mu_x$.
Now to show that $D \subseteq C(B)$, let $x \notin C(B)$.so $x \in X - C(B)$ and $X - C(B)$ is open.
i.e, $X - C(B)$ is a neighborhood of x.
Also $B \cap X - C(B) = \phi$.
Hence $x \notin D$.if $x \in D$, then).so $D \subseteq C(B)$for each $U \in \mu x$. and $U \cap B \neq \phi$. then $c \in C(B)$. $x \in C(B)$

9) Assume x∈I(B)
We have $I(B) = \cup \{U: U$ is open and $U \subseteq B\}$.
So $U \subseteq B$ for some $U \in \mu x$.
Conversely assume that $U \subseteq B$ for some $U \in \mu_x$ and $x \in B$.then
$x \in \{U: U$ is open and $U \subseteq B\}$. so $x \in I(B)$.

10) We have $d(B) = \{U \cap (B - \{0\}) \neq 0\ \forall x \in U and\ U \in \mu\}$
Let $B = \{x\}$. if $x \in U$ and $U \in \mu_x$.
Then $U \cap (B - \{x\}) = U \cap \phi = \phi$. So $x \notin B$.
Hence $x \notin d(B)$. this shows that $x \notin d(\{x\})$ for each $x \in X$.

Definition Let X be a generalized topological space, and μ_x = {U: U∈ μ , x∈U}.
1) Let x∈X and U∈ μ_x .then x is called the representative element of U, if U⊂V for each V∈ μ_x .
2) a space X is called a C_0-space if C_0=X where C_0 is the set of all representative elements of set of μ.
3) Let x∈X the set Md(x) ={ U∈ μ_x :U⊃V∈ μ_x⇒V=U} is called the minimal description of X.

Remark
Let X be a space and x∈X if μ_x is finite, then $x \notin C_0$ $|Md(x)| > 1$ in which $|Md(x)|$ is the cardinality of $Md(x)$.

Let $x \neq C_0$. that means x is not a representative element some open set V∈ μ.so the set $Md(x) = \{U \in \mu x : U \supset V$ and $U \neq V\}$

Hence $|Md(x)| > 1$.

Conversely suppose $|Md(x)| = 1$ then $\exists U, V \in \mu_x$. such that $U \supset V$ and $U = V$. i.e , $x \in U$ for some $U \in \mu$.so X is representative of U.this shows that

$x \in C_0$, which a contradiction is.∴ $|Md(x)| > 1$.

Proposition
Let B_1, B_2 be subsets of a C_0 space X. Then
1) $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$
2) $C(B_1 \cup B_2) = C(B_1) \cup C(B_2)$

Proof

1) We have $(B_1 \cap B_2) \subset B_1$ and $B_1 \cap B_2 \subset B_2$
Then we have $I(B_1 \cap B_2) \subset I(B_1)$ and $I(B_1 \cap B_2) \subset I(B_2)$.
So $I(B_1 \cap B_2) \subset I(B_1) \cap I(B_2)$ (1)

If $x \in I(B_1) \cap I(B_1)$

Then there are $U1, U2 \in \mu_x$ such that $U1 \subset B1$ and $U2 \subset B2$.

Since X is a a C_x space $x \in C_0$.

So there is $U \in \mu_x$ such that X is representative element of U and hence $U \subset U_1$ and $U \subset U_2$

Consequently $x \in U \subset U_1 \cap U_2 \subset B_1 \cap B_2$

It follows that $x \in I(B1 \cap B2)$.

Thus $I(B_1) \cap I(B_2) \subset I(B_1 \cap B_2)$. (2)

From (1) and (2) $I(B_1) \cap I(B_2) = I(B_1 \cap B_2)$

2) We have $B_1 \subset B_1$ and $B_2 \subset B_1 \cup B_2$.
Then $C(B_1) \subset C(B_1 \cap B_2)$ and $C(B2) \subset C(B_1 \cap B_2)$.
So $C(B_1) \cup C(B_2) \subset C(B_1 \cup B_2)$ (1)
If $x \in C(B_2 \cup B_2)$then $x \in X - I(X - B_1 \cup B_2)$.
So $x \in X - I(B_1^C \cap B_2^C)$
It follows that $x \in X - I(B_1^C) \cap I(B_2^C)$.
i.e., $x \in [X - I(B_1^C)] \cup [X - I(B_2^C)]$
so $x \in C(B_1) \cup C(B_2)$.
Hence $C(B_2) \cup C(B_2) \supset C(B_1 \cup B_2)$.
i.e., $C(B_1 \cup B_2) \subset C(B_1) \cup C(B_2)$. (2)
From (1) and (2) , $C(B_1 \cup B_2) \subset C(B_1) \cup C(B_2)$.

Proposition
Let X be a space. If μ is finite, then the following are equivalent.
1) X is a C_0-space.
2) $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$.
3) $C(B_1 \cup B_2) = C(B_1) \cup C(B_2)$ for each $B_1, B_2 \in P(X)$.where P(X) is the power set of X.

Proof

1⟹2
Let $B_1 \cap B_2 \subset B_1$and $B_1 \cap B_2 \subset B_2$

Then we have $I(B_1 \cup B_2) \subset I(B_1) \cap I(B_2)$, (1)

If $x \in I(B_1) \cap I(B_2)$, then there are $U_1, U_2 \in \mu_x$ such that $U_1 \subset B_1$ *and* $U_2 \subset B_2$

Since X is a C_0 space, $x \in C_0$.

So, there is $U \in \mu_x$ such that X is a representative element of U and hence $U \subset U_1$ and $U \subset U_2$,

Consequently $x \in U \subset U1 \cap U2 \subset B_1 \cap B_2$.

It follows that $x \in I(B_1 \cap B_2)$.

Thus $I(B_1.) \cap I(B_2.) \subset I(B_1 \cap B_2.)$ (2)

From (1) and (2) we get $I(B_1) \cap I(B_2) = I(B_1 \cap B_2)$ for each B_1, B_2∈P(X).

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2⇒ 1

If X is not a C_0 space, then there is $x \in X$ such that $x \notin C0$.then $|Md(x)| > 1$.

So, there are $U_1, U_1 \in Md(x)$ such that $U_1 \notin U_2$.

Hence $x \in U_1 \cap U_2 = I(U_1) \cap I(U_2)$.on the other hand for each $U \in \mu_x$, $U \not\subset U_1 \cap U_2$ because $U_1, U_2 \in Md(x)$.

So $x \notin I(U_1 \cap U_2)$

This contradicts $I(U_1 \cap U_2) = I(U_1) \cap I(U_2)$.

So X is a C_0 space.

2⇒3

Assume $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$.

Then $C(B_1 \cup B_2) = X - I(X - B_1 \cup B_2))$
$= X - I(B_1^C \cap B_2^C)$
$= X - [I(B_1^C) \cap I(B_2^C)$
$= [X - I(B_1^C)] \cup [X - I(B_2^C)]$
$= C(B_1) \cap C(B_2)$

So $C(B_1 \cup B_2) = C(B_1) \cap C(B_2)$.

1⇒2

Assume $C(B_1 \cup B_2) = C(B_1) \cap C(B_2)$then
$I(B_1 \cup B_2) = X - C(X - (B_1 \cap B_2)) = X - C(B_1^C \cap B_2^C) = X - [C(B_1^C) \cup C(B_2^C)]$
$= [X - C(B_1^C)] \cap [X - C(B_2^C)] = [X - C(X - B_1)] \cap [X - C(X - B_2)]$
$= I(B_1) \cup I(B_2)$
So $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$

Remark

Finite in the above proposition cannot be omitted. In fact let X be the closed interval [0,1].then $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$and $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$ for each $B_1, B_2 \in P(X)$.but X is not a C_0 space.

Base

Definition Let $\beta \subseteq P(X)$, then β is called a base for a generalized topology μ if $\mu = \{\cup\beta' : \beta' \subseteq \beta\}$.

Proposition

$\beta \subseteq P(X)$ is a base for a generalized topology μ ⇔whenever U is a μ-open set and $x \in U$, then $\exists B \in \beta \subseteq U$.

Proof

Let β be a base for μ.
$\therefore \mu = \{\cup \beta' : \beta' \subseteq \beta\}$.

If U is a μ-open set, then $U \in \mu$ and thus $\exists \beta' \subseteq \beta$ such that $U = \subset \beta'$.

Since $x \in U$, there is a $B' \in \beta'$.so that $x \in B', B' \subseteq \cup \beta' = U$.
$\therefore \boldsymbol{x} \in B' \subseteq U$.

Conversely let U be a set ,we show $\exists \beta' \subseteq \beta$. So that $U = \cup \beta'$.

By supposition for each $x \in U$ there is $Bx \subseteq U$.

Now if we consider $\beta' = \{B_x : x \in U\}$ it will be clear that $U = \cup \beta'$.

Theorem

β Is a base for some strong generalized topology $\Leftrightarrow X = \cup_{B \epsilon \beta} B$.

Proof

Let β is a base for some strong generalized topology μ.since x is a μ-open set, by above proposition $\forall x \in X, \exists\, a\, B_x \in \beta\ so\ that\ x \in B_x \subseteq U$.
$\therefore \boldsymbol{X} \subseteq \cup_{x \epsilon X} Bx \subseteq \cup_{B \epsilon \beta} B \subseteq X \Rightarrow X = \cup_{B \epsilon \beta} B$.

Conversely suppose that $X = \cup_{B \epsilon \beta} B$.

Consider the set $\mu = \{\cup \beta' : \beta' \subseteq \beta\}$.we show that is a strong generalized topology if we take $\beta' = \phi$,then $c' = \{\phi\}. \therefore \phi \in \mu$.

Any union of elements of μ again has a form like its elements and there for belongs to μ.on the other hand by definition it is obvious that μ is a base for μ.

Theorem

A subset of P(X) is a base for some generalized topology on X.

Proof

Let β *be a subset of* P(X).

Consider the set $\mu = \{\cup \beta' : \beta' \subseteq \beta\}$.

If we repeat the proof of the converse part in the above theorem, we conclude that μ is a generalised topology which has β as a base.

2. Conclusion

Transitioning from a standard to a generalised topology is relatively straightforward. It involves the inclusion of additional open sets, which enhances the structure of the set and introduces a more decadent array of properties. This discussion provides an introductory overview of generalised topological spaces, highlighting some fundamental results within this framework. Despite the progress made, numerous open problems still need to be solved, which presents a rich avenue for further exploration. Furthermore, various topological properties require new definitions when applied to generalised topological spaces, indicating a need for continued research. The field of generalised topology is vibrant and evolving, with recent studies contributing valuable insights, the specifics of which can be found in the referenced materials.

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