On Bc-open sets

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Abstract: In this paper, we introduce a new class of open sets, called Bc-open sets, it is denoted and studied. Also, we have studied of definition Bc-paracompact spaces and nearly Bc-paracompact spaces and have provide some properties of this concepts.

Keywords: 0-open, Bc-open

1.Introduction

In [5] H. Z. Ibrahim introduced the concept of Bc-open set in topological spaces. This paper consist of two sections. In section one, we give similar definition by using of Bc-open sets and also we proof some properties about it. In section two we obtain new a characterization and preserving theorems of Bc-paracompact spaces, nearly Bc-paracompact spaces and the product of space X X Y where X is Bc-paracompact space and Y is 0-compact space.

Definition(1.1)[3]:
Let X be a topological space and A ⊂ X. Then A is called b-open set in X if A ⊂ X. The family of all b-open subset of a topological space (X, τ) is denoted by BO(X, τ) or (Briefly BO(X)).

Definition(1.2)[5]:
Let X be a topological space and A ⊂ X. Then A is called Bc-open set in X if for each x ∈ A ∈ BO(X, τ), there exists a closed set F such that x ∈ F C A. The family of all Bc-open subset of a topological space (X, τ) is denoted by BcO(X, τ) or (Briefly BcO(X)), A is Bc-closed set if A ⊂ A C is Bc-open set. The family of all Bc-closed subset of a topological space (X, τ) is denoted by BcC(X, τ) or (Briefly BcC(X)).

Remark(1.3):
It is clear from the definition that every Bc-open set is b-open, but the converse is not true in general as the following example:

Let X = {1, 2, 3}, τ = {ϕ, X, {1}, {2}, {1, 2}}. Then the closed set are: X, ϕ, {2, 3}, {1, 3}, {3}. Hence BO(X) = {ϕ, X, {1}, {2}, {1, 2}, {1, 3}, {2, 3}} and BcO(X) = {ϕ, X, {1}, {2}, {3}}. Then {1} is b-open but {1} is not Bc-open.

Definition (1.4)[10]:
1) Let X be a topological space and A ⊂ X. Then A is called 0-open set in X if for each x ∈ A, there exists an open set G such that x ∈ G C C A. The family of all 0-open subset of a topological space (X, τ) is denoted by 0O(X, τ) or (Briefly 0O(X)).
2) Let X be a topological space and A ⊂ X. A point x ∈ X is said to 0-interior point of A, if there exist an 0-open set U such that x ∈ U C A. The set of all 0-interior points of A is called 0-interior of A and is denoted by A×0.
3) Let X be a topological space and A ⊂ X. The 0-closure of A is defined by the intersection of all Bc-closed sets in X containing A, and is denoted by A×0.

Remark (1.5)[5]:
1) Every 0-open is Bc-open.
2) Every 0-closed is Bc-closed.

Example (1.6):
The intersection of two Bc-open sets is not Bc-open in general. Let X = {1, 2, 3}, τ = {ϕ, X, {1}, {2}, {1, 2}}. Then {1, 3}, {2, 3} is Bc-open set where as {1, 3}∩{2, 3} = {3} is not Bc-open set.

Remark (1.7)[2]:
The intersection of an b-open set and an open set is b-open set.

Proposition (1.8):
Let X be a topological space and A, B ⊂ X. If A is Bc-open set and B is an 0-open set, then A ∩ B is Bc-open set.

Proof:
Let A be a Bc-open set and B is an 0-open set, then A is b-open set and B is an open set since every 0-open is open. Then A ∩ B is b-open set by (Remark(1.7)). Now, let x ∈ A ∩ B, x ∈ A and x ∈ B, then there exists a closed set F such that x ∈ F ⊂ A and there exists an open set E such that x ∈ E ⊂ E ⊂ B. Therefore, E ∩ E is closed since the intersection of closed sets is closed. Thus x ∈ E ∩ E ⊂ A ∩ B. Then A ∩ B is Bc-open set.

Proposition (1.9)[5]:
Let X be a topological space and A ⊂ X. Then A is Bc-open set if and only if A is b-open set and it is a union of closed sets. That is A = Uα Fα where A is b-open set and Fα is closed sets for each α.

Proposition (1.10)[5]:
Let {Aα; α ∈ A} be a collection of Bc-open sets in a topological space X. Then Uα Aα is Bc-open.

Lemma (1.11)[4]:
Let X be a topological space and Y ⊂ X. If G is an 0-open in X, then G ∩ Y is an 0-open in Y.

Proposition (1.12)[5]:
Let X be a topological space and Y ⊂ X. If G is a Bc-open set in X and Y is an open in X, then G ∩ Y is b-open in Y.
Proposition (1.13):
Let \( X \) be a topological space and \( Y \subseteq X \). If \( G \) is a \( \theta \)-open in \( X \) and \( Y \) is a \( \theta \)-open in \( X \), then \( G \cap Y \) is \( \theta \)-open in \( Y \).

Proof:
Let \( x \in G \cap Y \), \( x \in G \) and \( x \in Y \). Since \( G \) is a \( \theta \)-open set in \( X \), then for each \( x \in G \in BO(X) \), there exists \( F \) is closed set in \( X \) such that \( x \in F \in G \) and since \( Y \) is an \( \theta \)-open in \( X \), then there exists \( U \) open set in \( X \) such that \( x \in U \subseteq \overline{U} \subseteq Y \). Since \( G \) is \( \theta \)-open, then \( G \) is \( Bc \)-open and since \( Y \) is an \( \theta \)-open, then \( Y \) is an open set by proposition (1.12). Therefore, \( G \cap Y \) is \( \theta \)-open in \( Y \). Since \( F, U \) are closed set in \( X \) and \( Y \subseteq X \), then \( F \cap U \) is closed set in \( Y \). Thus \( x \in F \cap U \subseteq G \cap Y \). Hence \( G \cap Y \) is \( \theta \)-open in \( Y \).

Proposition (1.14):
Let \( X \) be a topological space and \( Y \) is an \( \theta \)-open subset of \( X \). If \( G \) is an \( \theta \)-open in \( Y \), then \( G \) is \( \theta \)-open in \( X \).

Proof:
Suppose that \( Y \) is an \( \theta \)-open subset of \( X \) and \( G \subseteq Y \), since \( G \) is a \( \theta \)-open set in \( Y \), then for each \( x \in G \in BO(Y) \), there exists \( F \) is closed set in \( Y \) such that \( x \in F \subseteq G \). Let \( G = Y \cup U \), \( U \subseteq Y \), and \( F = E \cap Y \subseteq X \). Then \( x \in E \subseteq Y \). Hence \( G \) is \( \theta \)-open in \( X \).

Lemma (1.15)[6]:
Let \( X \) and \( Y \) be a topological spaces and let \( A \subseteq X, B \subseteq Y \) be two non empty subset:
1) If \( A \) is an open set in \( X \) and \( B \) is an open set in \( Y \), then \( A \times B \) is an open set in \( X \times Y \).
2) If \( A \) is a closed set in \( X \) and \( B \) is a closed set in \( Y \), then \( A \times B \) is a closed set in \( X \times Y \).
3) \((A \times B) = A \times \overline{B}\).

Theorem (1.16):
Let \( X \) and \( Y \) be a topological spaces and let \( A \subseteq X, B \subseteq Y \) such that \( A \) is an \( \theta \)-open set of \( X \), \( B \) is an \( \theta \)-open set of \( Y \), then \( A \times B \) is an \( \theta \)-open subset of \( X \times Y \).

Proof:
Let \( A \) be an \( \theta \)-open set of \( X \) and \( B \) be an \( \theta \)-open set of \( Y \), then for each \( x \in A \), there exists \( G \) open set in \( X \) such that \( x \in G \subseteq \overline{G} \subseteq A \) and for each \( y \in B \), there exists \( U \) open set in \( X \) such that \( y \in U \subseteq \overline{U} \subseteq B \). By lemma (1.15)(1), then \( G \times U \) is an open set in \( X \times Y \). Since \( G \), \( U \) is closed set, then \( G \times U \) is a closed set in \( X \times Y \) by lemma (1.15)(2). Since \( G \times U = G \times \overline{U} \) by lemma (1.15)(3), then \( x \in G \times U \subseteq G \times \overline{U} \subseteq A \times X \). Hence \( A \times B \) is an \( \theta \)-open subset of \( X \times Y \).

Proposition (1.17)[8]:
Let \( X \) and \( Y \) be a topological spaces and let \( A \subseteq X, B \subseteq Y \) such that \( A \) is a \( Bc \)-open set of \( X \), \( B \) is an open set of \( Y \), then \( A \times B \) is a \( Bc \)-open subset of \( X \times Y \).

Proposition (1.18):
Let \( X \) and \( Y \) be a topological spaces and let \( A \subseteq X, B \subseteq Y \) such that \( A \) is a \( Bc \)-open set of \( X \), \( B \) is an \( \theta \)-open set of \( Y \), then \( A \times B \) is a \( Bc \)-open subset of \( X \times Y \).

Proof:
Let \( A \) be a \( Bc \)-open set of \( X \) and \( B \) be an \( \theta \)-open set of \( Y \), then for each \( x \in A \in BO(X) \), there exists \( F \) closed set in \( X \) such that \( x \in F \subseteq A \) and for each \( y \in B \), there exists \( U \) open set in \( Y \) such that \( y \in U \subseteq \overline{U} \subseteq B \). Since \( A \) is a \( Bc \)-open in \( X \) and \( B \) is an \( \theta \)-open in \( Y \), then \( A \) is a \( Bc \)-open in \( X \) and \( B \) is an open in \( Y \). Thus \( A \times B \) is a \( Bc \)-open subset of \( X \times Y \) by proposition (1.17), \( x \in A \) and \( y \in B \), then \((x,y) \in A \times B \in BO(X) \). Since \( x \in F \subseteq A \) and \( y \in U \subseteq \overline{U} \subseteq B \) such that \( F \) is closed set in \( X \) and \( U \) is closed set in \( Y \), then \( F \times U \) is closed set in \( X \times Y \). Therefore, \((x,y) \in F \times U \subseteq A \times B \). Hence \( A \times B \) is a \( Bc \)-open subset in \( X \times Y \).

Definition (1.19)[1]:
Let \( X \) be a topological space and \( x \in X \). Then a subset \( N \) of \( x \) is said to be a \( \theta \)-neighborhood of \( x \), if there exists \( \theta \)-open set \( U \) in \( X \) such that \( x \in U \subseteq N \).

Definition (1.20)[5]:
Let \( X \) be a topological space and \( A \subseteq X \). A point \( x \in X \) is said to be \( Bc \)-interior point of \( A \) if there exist a \( Bc \)-open set \( U \) such that \( x \in U \subseteq A \). The set of all \( Bc \)-interior points of \( A \) is called \( Bc \)-interior of \( A \) and is denoted by \( A^{Bc} \).

Theorem (1.21)[5]:
Let \( X \) be a topological space and \( A, B \subseteq X \), then the following statements are! true:
1) \( A^{Bc} \) is the union of all \( Bc \)-open set which are contained in \( A \).
2) \( A^{Bc} \) is \( Bc \)-open in \( X \).
3) \( A^{Bc} \subseteq A \).
4) \( A \) is \( Bc \)-open if and only if \( A = A^{Bc} \).
5) \( (A^{Bc})^{Bc} = A^{Bc} \).
6) If \( A \subseteq B \), then \( A^{Bc} \subseteq B^{Bc} \).
7) \( A^{Bc} \cup B^{Bc} = (A \cup B)^{Bc} \).
8) \( A \cap B^{Bc} = A^{Bc} \cap B^{Bc} \).

Definition (1.22)[5]:
Let \( X \) be a topological space and \( A \subseteq X \). The \( Bc \)-closure of \( A \) is defined by the intersection of all \( Bc \)-closed sets in \( X \) containing \( A \), and is denoted by \( \overline{A^{Bc}} \).

Theorem (1.23)[5]:
Let \( X \) be a topological space and \( A, B \subseteq X \). Then the following statements are true:
1) \( \overline{A^{Bc}} \) is the intersection of all \( Bc \)-closed sets containing \( A \).
2) \( \overline{A^{Bc}} \) is \( Bc \)-closed set in \( X \).
3) \( A \subseteq \overline{A^{Bc}} \).
4) \( A \) is \( Bc \)-closed set if and only if \( A = \overline{A^{Bc}} \).
5) \( (\overline{A^{Bc}})^{Bc} = \overline{A^{Bc}} \).
6) If \( A \subseteq B \), then \( \overline{A^{Bc}} \subseteq \overline{B^{Bc}} \).
7) \( \overline{A^{Bc}} \cup \overline{B^{Bc}} = \overline{(A \cup B)^{Bc}} \).
8) \( (A \cap B)^{Bc} \subseteq \overline{A^{Bc}} \cap \overline{B^{Bc}} \).

Proposition (1.24)[5]:
Let \( X \) be a topological space and \( A \subseteq X \). Then \( x \in \overline{A^{Bc}} \) if and only if \( A \cap U \neq \phi \) for every \( Bc \)-open set \( U \) containing \( x \).
Definition(1.25)[5]:
Let $X$ be a topological space and $A \subset X$. A point $x$ is said to be $Bc$-limit point of $A$, if for each $Bc$-open set $U$ containing $x$, $U \cap (A - \{x\}) \neq \emptyset$. The set of all $Bc$-limit points of $A$ is called a $Bc$-derived set of $A$ and is denoted by $A^{bc}$.

Proposition(1.26)[5]:
Let $X$ be a topological space and $A \subset X$. Then $A^{bc} = A \cup A^{bc}$.

Proposition(1.27):
Let $X$ be a topological space and $A \subset X$, then $A^{bc}$ is the smallest $Bc$-closed set containing $A$.

Proposition(1.28)[5]:
Let $X$ be a topological space and $A \subset X$, then the following statements are true:
1) $(A^{bc})^{c} = (A^{c})^{abc}$.
2) $(A^{+bc})^{c} = (A^{c})^{b}$.
3) $A^{bc} = (A^{c})^{b}$.
4) $A^{+bc} = (A^{c})^{c}$.

Definition(1.29):
Let $X$ be a topological space and $A \subset X$, $A$ is called 0-regular open set in $X$ iff $A = A^{0-\theta}$. The complement of 0-regular open set is called 0-regular closed.

Definition(1.30):
Let $X$ be a topological space and $A \subset X$, $A$ is called $Bc$-regular open set in $X$ iff $A = A^{bc-Bc}$. The complement of $Bc$-regular open set is called $Bc$-regular closed.

Remark(1.31):
Let $X$ be a topological space and $A \subset X$, $A$ is $Bc$-regular open set, then $A^{bc-Bc}$ is $Bc$-regular open set.

Proof:
To prove $A^{bc-Bc}$ is $Bc$-regular open we must prove that $A^{bc-Bc} = A^{Bc-Bc-Bc}$, since $A \subset A^{bc}$, then $A^{bc} \subset A^{Bc-Bc}$ and since $A$ is $Bc$-open set, hence $A \subset A^{Bc-Bc} A^{Bc-Bc} \subset A^{Bc-Bc-Bc} ... (1)$ Since $A^{Bc-Bc} \subset A^{Bc}$, then $A^{Bc-Bc} \subset A^{Bc-Bc-Bc}$ and $A^{Bc-Bc-Bc} = A^{Bc}$, hence $A^{Bc-Bc-Bc} \subset A^{Bc-Bc}$ ... (2) From (1) and (2) we get $A^{Bc-Bc} = A^{Bc-Bc-Bc}$. Hence $A^{BC-Bc}$ is $Bc$-regular open.

2. Separation Axiom

Definition(2.1)[7]:
A space $X$ is called $\partial X = \partial X - space$ iff for each $x \neq y$ in $X$ there exist disjoint 0-open sets $U, V$ such that $x \in U, y \in V$.

Definition(2.2):
A space $X$ is called $Bc$-regular space iff for each $x$ in $X$ and $C$ 0-closed set such that $x \in C$, there exist disjoint $Bc$-open sets $U, V$ such that $x \in U, C \subseteq V$. 

Proposition(2.3):
A space $X$ is $Bc$-regular space iff for every $x \in X$ and each 0-open set $U$ in $X$ such that $x \in U$ there exists a $Bc$-open set $W$ such that $W \subseteq W^{bc} \subseteq U$.

Proof:
Let $X$ be a $Bc$-regular space and $x \in X$, $U$ is 0-open in $X$ such that $x \in U$. Thus $U^{c}$ is 0-closed set, $x \in U^{c}$. Then there exist disjoint $Bc$-open set $W, V$ such that $x \in W, U^{c} \subseteq V$. Conversely let $F$ be an 0-open set such that $x \notin F$. Then $F^{c}$ is an 0-open set and $x \notin F^{c}$. Thus there exist $W$ be $Bc$-open set such that $x \in W \subseteq W^{bc} \subseteq F^{c}$. Then $x \in W, F \subseteq (W^{bc})^{c}$ and $W, (W^{bc})^{c}$ are disjoint $Bc$-open sets. Hence $X$ is $Bc$-regular space.

Definition(2.4):
A space $X$ is called $Bc^{*}$-regular space iff for each $x \in X$ and $Bc$-closed set $C$ such that $x \notin C$, there exist disjoint sets $U, V$ such that $U$ is 0-open, $V$ is a $Bc$-open and $x \in U, C \subseteq V$.

Proposition(2.5):
A space $X$ is $Bc^{*}$-regular space iff for every $x \in X$ and each $Bc$-open set $U$ in $X$ such that $x \in U$ there exists an 0-open set $W$ such that $x \in W \subseteq W^{bc} \subseteq U$.

Proof:
Let $X$ be a $Bc^{*}$-regular space and $x \in X, U$ be $Bc$-open in $X$ such that $x \in U$. Thus $U^{c}$ is $Bc$-closed set, $x \notin U^{c}$. Then there exist disjoint $Bc$-open set $W, V$ such that $W \subseteq \emptyset, V^{c} \subseteq U$. Conversely let $F$ be a $Bc$-closed set such that $x \in F$. Then $F^{c}$ is a 0-open set and $x \in F^{c}$. Thus there exist $W$ be $Bc$-open set such that $x \in W \subseteq W^{bc} \subseteq F^{c}$. Then $x \in W, F \subseteq (W^{bc})^{c}$ and $W, (W^{bc})^{c}$ are disjoint $Bc$-open sets. Hence $X$ is $Bc^{*}$-regular space.

Definition(2.7):
A space $X$ is called almost $Bc$-regular space iff for each $x$ in $X$ and $C$ is 0-closed regular closed set such that $x \notin C$, there exist disjoint $Bc$-open sets $U, V$ such that $x \in U, C \subseteq V$.

Definition(2.8):
A space $X$ is called almost $Bc^{*}$-regular space iff for each $x$ in $X$ and $C$ is $Bc$-regular closed set such that $x \notin C$, there exist disjoint $Bc$-open sets $U, V$ such that $U$ is 0-open, $V$ is $Bc$-open and $x \in U, C \subseteq V$.

Proposition(2.9):
A space $X$ is almost $Bc$-regular space iff for every $x \in X$ and each 0-closed set $U$ in $X$ such that $x \in U$ there exists an $Bc$-open set $W$ such that $x \in W \subseteq W^{bc} \subseteq U$.

Proof:
Let $X$ be a almost $Bc$-regular space and $x \in X, U$ is 0-closed in $X$ such that $x \in U$. Thus $U^{c}$ is 0-closed set, $x \notin U^{c}$. Then there exist disjoint $Bc$-open set $W, V$ such that $x \in W, U^{c} \subseteq V$. Conversely let $F$ be an 0-closed set such that $x \notin F$. Then $F^{c}$ is an 0-closed set and $x \notin F^{c}$. Thus there exist $W$ be $Bc$-open set such that $x \in W \subseteq W^{bc} \subseteq F^{c}$. Then $x \in W, F \subseteq (W^{bc})^{c}$ and $W, (W^{bc})^{c}$ are disjoint $Bc$-open sets. Hence $X$ is $Bc$-regular space.
A space $X$ is almost $Bc$-regular space iff for every $x \in X$ and each $Bc$-regular open set $U$ in $X$ such that $x \in U$ there exists an $0$-open set $W$ such that $x \in W \subseteq \overline{W}^{Bc} \subseteq U$.

**Proof:**
Let $X$ be a $Bc$-normal space and $x \in X$ be $Bc$-regular open in $X$ such that $x \in U$. Thus $U^c$ is $Bc$-closed set, $x \notin U^c$. Then there exist disjoint set $W, V$ such that $W$ is an $0$-open, $V$ is a $Bc$-open and $x \in W, U^c \subseteq V$. Hence $x \in W \subseteq \overline{W}^{Bc} \subseteq U^c \subseteq V$. Conversely, let $F$ be an $Bc$-regular closed set such that $x \notin F$. Then $F^c$ is an $Bc$-regular open set and $x \in F^c$. Thus there exist $W$ is $0$-open set such that $x \in W \subseteq \overline{W}^{Bc} \subseteq F^c$. Then $x \in W, F \subseteq (\overline{W}^{Bc})^c$ and $(\overline{W}^{Bc})^c$ is $Bc$-open set, $W \cap (\overline{W}^{Bc})^c = \emptyset$. Hence $X$ is almost $Bc$-regular space.

**Definition 2.11:**
A space $X$ is called $Bc$-normal space iff for every disjoint $0$-closed set $F_1, F_2$ there exist disjoint $Bc$-open sets $V_1, V_2$ such that $F_1 \subseteq V_1, F_2 \subseteq V_2$.

**Proposition 2.12:**
A space $X$ is called $Bc$-normal space iff for every $0$-closed set $F \subseteq X$ and each $0$-open set $U$ in $X$ such that $F \subseteq U$ there exists an $0$-open set $W$ such that $F \subseteq W \subseteq \overline{W}^{Bc} \subseteq U$.

**Proof:**
Let $X$ be a $Bc$-normal space and $F$ is an $0$-closed set in $X, U$ is an $0$-open set such that $F \subseteq U$. Thus $U^c$ is $0$-closed set $U^c, F$ are disjoint $0$-open set, there exists $0$-closed set $W, V$ such that $F \subseteq W$, $U^c \subseteq V, W \cap V = \emptyset$. Hence $F \subseteq W \subseteq \overline{W}^{Bc} \subseteq \overline{W}^{Bc} \subseteq U^c \subseteq V \subseteq U$. Conversely, let $F_1, F_2$ be a disjoint $0$-closed set. Then $F_1^c$ is an $0$-open set and $F_2 \subseteq F_2^c$. Thus there exist $W$ is $Bc$-open set such that $F_1 \subseteq W \subseteq \overline{W}^{Bc} \subseteq F_2$. Then $F_1 \subseteq W, F_2 \subseteq (\overline{W}^{Bc})^c$ and $(\overline{W}^{Bc})^c$ is disjoint $Bc$-open set. Hence $X$ is $Bc$-normal space.

**Proposition 2.13:**
If $X$ is both $Bc$-normal and $\partial F - space$, then $X$ is $Bc$-regular.

**Proof:**
Let $x \in X$ and $U$ be an $0$-open set such that $x \in U$. Then $\{x\}$ is a $0$-closed subset of $X$. Thus there exists a $Bc$-open set $W$ such that $\{x\} \subseteq W \subseteq \overline{W}^{Bc} \subseteq U$. By proposition (2.12), $x \in W \subseteq \overline{W}^{Bc} \subseteq U$ and hence by proposition (2.3) $X$ is $Bc$-regular space.

### 3.Bc-paracompact Spaces

**Definition 3.1:**
A covering of a topological space $X$ is the family $\{A_\alpha; \alpha \in \Lambda\}$ of subsets of $X$ such that $\bigcup_{\alpha \in \Lambda} A_\alpha = X$. If each $A_\alpha$ is open, then $\{A_\alpha; \alpha \in \Lambda\}$ is called an open covering, and if each set $A_\alpha$ is closed, then $\{A_\alpha; \alpha \in \Lambda\}$ is called a closed covering. A covering $\{B_\gamma; \gamma \in \Gamma\}$ is said to be a refinement of a covering $\{A_\alpha; \alpha \in \Lambda\}$ if for each $\gamma$ in $\Gamma$ there exists some $\alpha$ in $\Lambda$ such that $B_\gamma \subseteq A_\alpha$.

**Definition 3.2:**
The family $\{B_\alpha; \alpha \in \Lambda\}$ of a subset of a space $X$ is said to be a $0$-locally finite if for each $x \in X$ there exist an $0$-neighborhood $N_x$ of $x$ such that the set $\{\alpha \in \Lambda : N_x \cap B_\alpha \neq \emptyset\}$ is finite.

**Proposition 3.3:**
If $\{B_\alpha; \alpha \in \Lambda\}$ is an $0$-locally finite family of subset of a space $X$, then there exist a family $\{C_\alpha; \alpha \in \Lambda\}$ of $Bc$-closed for each $\alpha$, then $\{C_\alpha; \alpha \in \Lambda\}$ is an $0$-locally finite.

**Proof:**
Let $\{B_\alpha; \alpha \in \Lambda\}$ is an $0$-locally finite, for each $x \in X$, then there exist $B_{\alpha_i}$ $0$-open set containing $x$ such that $B_{\alpha_i} \cap B_{\alpha_j} \neq \emptyset, i = 1, \ldots, n$. Hence $B_{\alpha_i} \cap B_{\alpha_j} = \emptyset, i = 1, n + 1, n + 2, \ldots$. Hence $\{\alpha_i; B_{\alpha_i} \subseteq \emptyset, i = 1, n + 1, n + 2, \ldots\}$ is an $0$-locally finite.

**Proposition 3.4:**
Let $(X, \tau)$ be a topological space and $A \subseteq X$. If $A_\alpha$ is an $0$-locally finite, then $\overline{A_\alpha}^{Bc}$ is an $0$-locally finite.

**Proof:**
Let $\{A_\alpha; \alpha \in \Lambda\}$ is an $0$-locally finite, for each $x \in X$, then there exist $0$-open set $G_x$ containing $x$ such that $G_x \cap A_\alpha \neq \emptyset, i = 1, \ldots, n$. Hence $G_x \cap A_\alpha = \emptyset, i = 1, n + 1, n + 2, \ldots$. Hence $G_x \cap A_\alpha = \emptyset, i = 1, 2, \ldots$. Then $G_x \cap \overline{A_\alpha}^{Bc} = \emptyset, i = 1, n + 1, n + 2, \ldots$ . Hence $\overline{G_x \cap A_\alpha}^{Bc} = \emptyset, i = 1, 2, \ldots$. Therefore, $\overline{G_x \cap A_\alpha}^{Bc}$ is an $0$-locally finite.

**Proposition 3.5:**
Let $\{A_\alpha; \alpha \in \Lambda\}$ is an $0$-locally finite $Bc$-closed family of a space $X$ then $\overline{A_\alpha}^{Bc} = \bigcup_{\alpha \in \Lambda} A_\alpha^{Bc}$.

**Proof:**
Since $A_\alpha \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha$, then $\overline{A_\alpha}^{Bc} \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha^{Bc}$ by theorem (1.23) and hence $\overline{A_\alpha}^{Bc} \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha^{Bc}$. To prove that $\bigcup_{\alpha \in \Lambda} A_\alpha^{Bc} \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha^{Bc}$. Let $x \in \bigcup_{\alpha \in \Lambda} A_\alpha^{Bc}$. Since $\{A_\alpha; \alpha \in \Lambda\}$ is an $0$-locally finite, then there exists an $0$-open set $G_x$ containing $x$ such that $G_x \cap A_\alpha \neq \emptyset$ for only a finite member of $\alpha$ say $\alpha_1, \ldots, \alpha_n$. Since $x \notin \bigcup_{\alpha \in \Lambda} A_\alpha^{Bc}$ for each $\alpha \in \Lambda$, then $x \notin A_\alpha^{Bc}$ and $x \notin A_\alpha$ for each $\alpha \in \Lambda$ by proposition (1.27). Thus there exists an $0$-open set $U_x$ which contain $x$ such that $U_x \cap A_\alpha = \emptyset$ for each $\alpha \neq \alpha_1, \ldots, \alpha_n$. Let $x \in U_x \cap A_\alpha = \emptyset$. Then $x \notin \bigcup_{\alpha \in \Lambda} A_\alpha^{Bc}$ and hence $\overline{A_\alpha}^{Bc} = \bigcup_{\alpha \in \Lambda} A_\alpha^{Bc}$.

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then $V \cap A_\alpha = \phi$, for $\alpha \in \Lambda$. Now, we have $\cap (\bigcup_{\alpha \in \Lambda} A_\alpha) = \phi$, so that since $x \in V$, then $x \notin \bigcup_{\alpha \in \Lambda} A_\alpha$, by proposition (1.24) which is a contradiction. Thus $\in \bigcup_{\alpha \in \Lambda} \bar{A}_\alpha = \bigcup_{\alpha \in \Lambda} \bar{A}_\alpha$, so that $\bar{A}_\alpha \cap \bar{A}_\beta \subseteq \bar{A}_\alpha$, then $\bigcup_{\alpha \in \Lambda} \bar{A}_\alpha = \bigcup_{\alpha \in \Lambda} \bar{A}_\alpha$.

Proposition (3.6): The union of member of 0-locally finite Bc-closed sets is Bc-closed. Proof: Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of 0-locally finite Bc-closed sets. Then $\bigcup_{\alpha \in \Lambda} \bar{A}_\alpha = \bigcup_{\alpha \in \Lambda} \bar{A}_\alpha = \bigcup_{\alpha \in \Lambda} A_\alpha$, by theorem (3.4) and hence $\bigcup_{\alpha \in \Lambda} A_\alpha$ is Bc-closed by theorem (1.23).

Theorem (3.7): Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of Bc-open subsets of a space $X$ and let $\{B_\beta\}_{\beta \in \Gamma}$ be an 0-locally finite Bc-closed covering of $X$ such that for each $\gamma \in \Gamma$ the set $\{\alpha \in \Lambda: B_\beta \cap A_\alpha \neq \phi\}$ is a finite. Then there exists 0-locally finite family $\{G_\alpha\}_{\alpha \in \Lambda}$ of Bc-open set of $X$ such that $A_\alpha \subseteq G_\alpha$ for each $\alpha \in \Lambda$.

Proof: For each $\alpha$, let $G_\alpha = \left(\{F_\gamma - B_\beta \cap A_\alpha = \phi\}\right)^c$. Clearly $A_\alpha \subseteq G_\alpha$ and since $\{B_\beta\}_{\beta \in \Gamma}$ is an 0-locally finite, it follow that $G_\alpha$ is Bc-open by proposition (3.6). Let $x$ be a point of $X$, there exists an 0-neighborhood $N_x$ of $x$ and a finite subset $k$ of $\Gamma$ such that $N_x \cap F_\gamma = \phi$ if $\gamma \in k$. Hence $\subseteq \bigcup_{\gamma \in k} F_\gamma$. Now $F_\gamma \cap A_\alpha \neq \phi$ iff $F_\gamma \cap A_\alpha \neq \phi$. For each $\alpha \in k$ the set $\{\alpha \in \Lambda: F_\gamma \cap A_\alpha \neq \phi\}$ is a finite. Hence $\{\alpha \in \Lambda: N \cap A_\alpha \neq \phi\}$ is a finite.

Lemma (3.8): If every 0-open cover of a topological space $X$ has an 0-locally finite Bc-closed refinement, then every 0-open cover of $X$ has an 0-locally finite Bc-closed refinement.

Proof: Let $U$ be 0-open cover of $X$, and $A = \{A_\alpha: s \in S\}$ an 0-locally finite of $U$ and for each $x \in X$ choose an 0-neighborhood $V_x$ of $x$ which meets only finitely many members of $A$. Let $F$ be an 0-locally finite Bc-closed refinement of the 0-open cover $\{V_x: x \in X\}$ and for each $s \in S$, let $W_s = \{(F \cap (F \cap A_\alpha)^c) \cap A_\alpha\}$, then $W_s$ is a Bc-open and contain $A_\alpha$, for each $s \in S$ and $F \in F$, we have $W_s \cap F \neq \phi$ iff $A_\alpha \cap F \neq \phi$. For each $s \in S$ take a $U_\gamma \subseteq U$ such that $A_\alpha \subseteq U_\gamma$ and let $U_s = W_s \cap U_\gamma$. The family $\{V_x\}_{x \in X}$ is a Bc-open refinement of $U$. Since for each $x \in X$ has an 0-neighborhood which meets only finitely many members of $F$ and every members of $F$ meets only finitely many members of $A$. Therefore, $\{V_x\}_{x \in X}$ is an 0-locally finite Bc-closed refinement of $U$.

Theorem (3.9): If every finite 0-open covering of a space $X$ has an 0-locally finite Bc-closed refinement, then $X$ is Bc-normal space.

Proof: Let $X$ be a topological space such that each finite 0-open covering of $X$ has an 0-locally finite Bc-closed refinement and let $A_\alpha \cap B \subseteq \bigcup_{\alpha \in \Lambda} \{A_\alpha \cap B\}$. The 0-open covering $\{A_\alpha \cup B\}$ of $X$ has an 0-locally finite Bc-closed refinement. Let $E$ be the union of the members of $W$ disjoint from $A$ and let $S$ be the union of the members of $W$ disjoint from $B$. Then $E$ and $S$ are Bc-closed sets and $E \cup \bar{S} = \bar{X}$. Thus if $G = (E)^c$ and $U = (S)^c$, then $G, U$ are disjoint Bc-open sets such that $A \subseteq G, B \subseteq U$. Hence $X$ is Bc-normal space.

Definition (3.10): A topological space $X$ is said to be Bc-paracompact if every 0-open covering of $X$ has an 0-locally finite Bc-open refinement.

Proposition (3.11): Let $X$ be a Bc-paracompact space, let $A$ be an 0-open subset of $X$ and let $B$ be an 0-closed set of $X$ which is disjoint from $A$. If for every $x \in B$ there exist 0-open sets $U_x$, $V_x$ such that $A \subseteq U_x$, $x \in V_x$ and $U_x \cap V_x = \phi$, then also there exist Bc-open sets $U, V$ such that $A \subseteq U$, $x \in V$ and $U \cap V = \phi$.

Proof: The family $\{V_x: x \in B\} \cup \{(B)^c\}$ is an 0-open cover of $B$-paracompact, so that it has an 0-locally finite Bc-open refinement $\{W_y\}_{y \in \Gamma}$. Let $\Gamma = \{y \in \Gamma: W_y \cap V_x < \text{for some } x \in B\}$. If $\Gamma = \emptyset$, then $U_y \cap W_y = \phi$ for some $x \in V$ by proposition (3.4), then $W_y^{bc}$ is an 0-locally finite Bc-closed. Therefore $A \cap W_y^{bc} = \phi$. Now, let $U = \left(\bigcup_{y \in \Gamma} W_y^{bc}\right)^c$ and $V = \bigcup_{y \in \Gamma} W_y$. Then $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.

Proposition (3.12): If $X$ is a Bc-paracompact $\theta T_2$-space, then $X$ is Bc-regular.

Proof: Let $x \in \text{X}$ and $F$ be an 0-closed set in $X$ such that $x \notin F$. Then for each $y \in F$ there exists 0-open sets $U_y$, $V_y$ such that $x \notin U_y$, $y \in V_y$. It follow from proposition (3.11) there exists Bc-open sets $U$ and $V$ such that $x \subseteq U$, $F \subseteq V$ and $U \cap V = \phi$. Thus $X$ is Bc-regular.

Proposition (3.13): Let $X$ be a topological space. If each 0-open covering of $X$ has an 0-locally finite Bc-closed refinement, then $X$ is Bc-paracompact Bc-normal Space.

Proof: Let $U$ be an 0-open covering of $X$ and let $\{A_\alpha\}_{x \in \Lambda}$ be an 0-locally finite Bc-closed refinement of $X$. Since $A_\alpha \subseteq \Lambda$, is an 0-locally finite, for each point $x \in X$ has an 0-neighborhood $G_x$ such that $\{\alpha \in \Lambda: G_x \cap \alpha \neq \phi\}$ is a finite. If $\{B_\beta\}_{\beta \in \Gamma}$ is an 0-locally finite Bc-closed refinement of the 0-open covering $\{G_\alpha\}_{x \in X}$, then for each $\gamma \in \Gamma$ the set $\{\alpha \in \Lambda: B_\beta \cap \alpha \neq \phi\}$ is a finite. It follows from theorem (3.9), that there exist an 0-locally finite family $\{A_\alpha\}_{x \in \Lambda}$ of Bc-open sets, such that $A_\alpha \subseteq \Lambda$ for each $x \in X$ and $A_\alpha$ be a member of $U$ such that $A_\alpha \subseteq \Lambda$ for each $\alpha \in \Lambda$.
Then \((V_{\alpha}\cap U_{\alpha})_{\alpha\in A}\) is an \(\theta\)-locally finite Bc-open refinement of \(X\). Thus \(X\) is Bc-paracompact, so that \(X\) is Bc-normal space by theorem(3,9).

**Theorem(3.14):**

Bc*-regular space is Bc-paracompact Bc-normal if and only if each \(\theta\)-open covering has an \(\theta\)-locally finite Bc-closed refinement.

**Proof:**

Suppose that \(X\) is Bc-paracompact Bc-normal space and let \((A_{\alpha})_{\alpha\in A}\) be an \(\theta\)-open covering of \(X\). Since \(X\) is Bc*-regular, there exists an \(\theta\)-open set \(V_{x}\) such that \(x \in V_{x} \subseteq \overline{V_{x}}^{\text{Bc}} \subseteq A_{\alpha}\) for some \(\alpha\). The family \((A_{\alpha}: x \in X)\) is an \(\theta\)-open cover of \(X\) and since \(X\) is Bc-paracompact, then there exists an \(\theta\)-locally finite Bc-open refinement \(W = \{W_{x}: x \in X\}\) of \((A_{\alpha}: x \in X)\). Hence \(\overline{W_{x}}^{\text{Bc}} \subseteq \overline{V_{x}}^{\text{Bc}} \subseteq A_{\alpha}\), then \(\{\overline{W_{x}}^{\text{Bc}}: x \in X\}\) is an \(\theta\)-locally finite Bc-open refinement of \((A_{\alpha})_{\alpha\in A}\). Conversely, from theorem(3,13).

**Theorem(3.15):**

Let \(X\) be any Bc*-regular space , the following condition are equivalent:

1) \(X\) is Bc-paracompact.
2) Every \(\theta\)-open cover of \(X\) has an \(\theta\)-locally finite refinement.
3) Every \(\theta\)-open cover of \(X\) has a Bc-closed \(\theta\)-locally finite refinement.

**Proof:**

1\(\rightarrow\)2

Let \(X\) be a Bc-paracompact space, then every \(\theta\)-open cover of \(X\) has an \(\theta\)-locally finite refinement.

2\(\rightarrow\)3

Let \(U\) be an \(\theta\)-open covering of \(X\). Since \(X\) is Bc*-regular, there exists an \(\theta\)-open set \(V_{x}\) such that \(x \in V_{x} \subseteq \overline{V_{x}}^{\text{Bc}} \subseteq U_{\alpha}\). The family \(V = \{V_{x}: x \in X\}\) is an \(\theta\)-open cover of \(X\), by (2) \(V\) has an \(\theta\)-locally finite refinement. Hence \(\{\overline{V_{x}}^{\text{Bc}}: x \in X\}\) is an \(\theta\)-locally finite Bc-open refinement of \(U\).

3\(\rightarrow\)1

By lemma(3,14).

**Lemma(3,16):**

Let \(X\) be any Bc*-regular Bc-paracompact space. Then every Bc-open cover \((G_{s}: s \in S)\) has an \(\theta\)-locally finite Bc-open refinement \((U_{s}: s \in S)\) such that \(\overline{U_{s}}^{\text{Bc}} \subseteq G_{s}\) for each \(s \in S\).

**Proof:**

Let \((G_{s}: s \in S)\) be any Bc-open cover of \(X\). For \(x \in X, x \in G_{s}\), for some \(s \in S\) and since \(X\) is Bc*-regular, hence by proposition(3,16), there exists an \(\theta\)-open cover \(W = \{W_{x}: x \in X\}\) and \(\overline{W_{x}}^{\text{Bc}} \subseteq G_{s}\). Since \(X\) is Bc-paracompact, then \(W\) has an \(\theta\)-locally finite Bc-open refinement \((A_{h}: h \in H)\) for each \(h \in H\) choose \(s(h) \in S\) such that \(\overline{A_{h}}^{\text{Bc}} \subseteq G_{s(h)}\) and let \(U_{s} = \overline{U_{s}}^{\text{Bc}} \subseteq \overline{A_{h}}^{\text{Bc}} \subseteq G_{s}\). Since \(s(h) := A_{h} \subseteq \overline{U_{s}}^{\text{Bc}} = \overline{U_{s}}^{\text{Bc}} \subseteq \overline{A_{h}}^{\text{Bc}} \subseteq G_{s}\), then \((U_{s}: s \in S)\) is an \(\theta\)-locally finite Bc-open refinement of \((G_{s}: s \in S)\) such that \(\overline{U_{s}}^{\text{Bc}} \subseteq G_{s}\) for each \(s \in S\).

**Definition(3,17):**

Let \(X\) be a topological space and \(A \subseteq X\). \(A\) is said to be Bc-density set if \(\overline{A}^{\text{Bc}} = X\).

**Definition(3,18):**

A topological space \(X\) is said to be Bc-Lindelof if every Bc-open cover of \(X\) has a countable sub cover.

**Theorem(3,19):**

Let \(X\) be any Bc*-regular Bc-paracompact space such that there exists an \(\theta\)-open Bc-dense Bc-Lindelof set , then \(X\) is a Bc-Lindelof space.

**Proof:**

Let \(U = \{U_{x}: x \in S\}\) be any Bc-open cover of \(X\). For each \(x \in X, x \in U_{x}\), for some \(s \in S\). By lemma(3,16), there exists a Bc-open \(\theta\)-locally finite refinement \((V_{x}: s \in S)\) of \(U\) such that \(\overline{V_{x}}^{\text{Bc}} \subseteq U_{x}\), for each \(s \in S\). Then \(\{V_{x}(A): s \in S\}\) is Bc-open cover of \(X\), by proposition(1,13). Since \(A\) is Bc-Lindelof , there exists a countable set \(S_{\alpha} \subseteq S\) such that \(A = \bigcup_{s \in S_{\alpha}} V_{s}\). Hence \(X = \overline{A}^{\text{Bc}} = \overline{\bigcup_{s \in S_{\alpha}} V_{s}}^{\text{Bc}} = \bigcup_{s \in S_{\alpha}} \overline{V_{s}}^{\text{Bc}} \subseteq \bigcup_{s \in S_{\alpha}} U_{x},\) hence \(X\) is Bc-Lindelof.

**Lemma(3,20):**

If \(U\) is an \(\theta\)-open covering of a topological space product \(X \times Y\) of a Bc-paracompact space \(X\) and an \(\theta\)-compact space , then \(U\) has a refinement of the form \(\{V_{x}^{\alpha} \times G_{\alpha}: i = 1,\ldots, n_{a}\}\), where \(\{G_{\alpha}: \alpha \in A\}\) is an \(\theta\)-locally finite Bc-open covering of \(X\), and for each \(\alpha\), \(G_{\alpha} = \{G_{\alpha}(i) = 1,\ldots, n_{a}\}\) is a finite \(\theta\)-open covering of \(Y\).

**Proof:**

Let \(x\) be a point of \(X\). Since \(Y\) is an \(\theta\)-compact there exists an \(\theta\)-open neighborhood \(W_{x}\) of \(x\) and a finite \(\theta\)-open covering \(G_{\alpha}\) of \(Y\) such that \(W_{x} \times G_{\alpha}\) is contained in some member of \(U\) if \(G_{\alpha} \in G_{\alpha}\). Let \(\{V_{x}^{\alpha}: \alpha \in A\}\) be an \(\theta\)-locally finite Bc-open refinement of open covering \(\{W_{x}^{\alpha}: x \in X\}\) of the Bc-paracompact space \(X\). For \(\alpha \in A\) choose \(x \in X\) such that \(V_{x}^{\alpha} \subseteq W_{x}^{\alpha}\) and let \(G_{\alpha} = \{G_{\alpha}(i) = 1,\ldots, n_{a}\}\). Then \(\{V_{x}^{\alpha}\}\) is a Bc-open refinement of \(U\).

**Proposition(3,21):**

The product of a Bc-paracompact space and an \(\theta\)-compact space is a Bc-paracompact space.

**Proof:**

Let \(X\) be a Bc-paracompact space and \(Y\) be an \(\theta\)-compact space and let \(U\) be an \(\theta\)-open covering of the topological product \(X \times Y\). Then by lemma(3,20) \(U\) has a Bc-open refinement of the form \(\{V_{x}^{\alpha} \times G_{\alpha}(i): i = 1,\ldots, n_{a}\}\), where \(\{G_{\alpha}(i): i \in A\}\) is an \(\theta\)-locally finite Bc-open refinement of \(Y\), for each \(i \in A\). Therefore, \(X \times Y\) is a Bc-paracompact space.

**Definition(3,22):**

A space \(X\) is said to be nearly Bc-paracompact space if each \(\theta\)-regular open covering of \(X\) has an \(\theta\)-locally finite Bc-open refinement.
Lemma (3.23):
Let $X$ be any almost $Bc^*$-regular nearly $Bc$-paracompact space. Then every $Bc$-regular open cover $\{G_s : s \in S\}$ has an $\theta$-locally finite $Bc$-regular open refinement $\{V_s : s \in S\}$ such that $V_s \subseteq G_s$ for each $s \in S$.

Proof:
Let $\{G_s : s \in S\}$ be any $Bc$-regular open cover of $X$. For $x \in X, x \in G_s$, for some $s \in S$ and since $X$ is almost $Bc^*$-regular, hence by proposition (2.10), there exists an $\theta$-regular open cover $W = \{W_x : x \in X\}$ and $W_x^{BC} \subseteq G_s$. Since $X$ is nearly $Bc$-paracompact, then $W$ has an $\theta$-locally finite $Bc$-open refinement $\{A_h : h \in H\}$ for each $h \in H$ choose $s(h) \in S$ such that $A_h^{BC} \subseteq G_{s(h)}$ and let $U_s = \bigcup_{s(h)=s} A_h$. Since $U_{s(h)=s} = \bigcup_{s(h)=s} A_h^{BC} = \bigcup_{s(h)=s} A_h^{BC} \subseteq G_s$, then $U_s \subseteq \bigcup_{s(h)=s} U_{s(h)=s}^{BC}$, hence $U_s \subseteq \bigcup_{s(h)=s} U_{s(h)=s}^{BC} \subseteq \bigcup_{s(h)=s} U_{s(h)=s}^{BC} \subseteq G_{s(h)}$. Let $V_s = \bigcup_{s(h)=s} U_{s(h)=s}^{BC}$, then $\{V_s : s \in S\}$ is an $\theta$-locally finite $Bc$-regular open refinement of $\{G_s : s \in S\}$ such that $V_s^{BC} \subseteq G_s$ for each $s \in S$.

Theorem (3.24):
For any space $X$, the following are equivalent:
1) $X$ is nearly $Bc$-paracompact.
2) Every $\theta$-regular open cover of $X$ has a $Bc$-regular open $\theta$-locally finite refinement.
3) Every $\theta$-regular open cover of $X$ has a $Bc$-regular closed $\theta$-locally finite refinement.

Proof:
1$\Rightarrow$2
Let $\mathcal{U}$ be any $\theta$-regular open cover of $X$, then $\mathcal{U}$ has an $\theta$-locally finite $Bc$-open refinement $\mathcal{V}$. Consider the family $\mathcal{W} = \{\bigcup_{s(h)=s} U_{s(h)=s}^{BC} : V \in \mathcal{V}\}$ is an $\theta$-locally finite $Bc$-regular open refinement of $\mathcal{U}$.
2$\Rightarrow$3
It is clear since every $Bc$-regular open set is $Bc$-regular closed set.
3$\Rightarrow$1
From lemma (3.8).

References