Some Fixed Point Theorems on Intuitionistic Fuzzy Metric Spaces with Implicit Relation

Ram Milan Singh[1], A. D. Singh[2], R. S. Chandel[3]

Abstract: ***In this paper, we give some fixed point theorems on intuitionistic fuzzy metric spaces with an implicit relation.***

Keyword: Triangular norm, triangular co-norm, intuitionistic fuzzy metric space, fuzzy metric space, fixed point

1.1 Introduction and Preliminaries

The concept of fuzzy sets was introduced by Zadeh [6] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [1], Erceg [2], Kaleva and Seikkala [4], Kutukcu [5], Kramosil and Michalek [3] have introduced the concept of fuzzy metric spaces in different ways. In this paper we give an implicit relation on intuitionistic fuzzy metric spaces and present some fixed point theorems given by some authors. Now, we begin with some definations.

1.2 Preliminaries

Definition 1.1 A binary operation $*: [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(a) $*$is commutative and associative;

(b) $*$is continuous;

(c) $a * 1 = a$ for all $a \in [0,1]$;

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Definition 1.2 A binary operation $\blacklozenge: [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-conorm if it satisfies the following conditions:

(a) $\blacklozenge$ is commutative and associative;

(b) $\blacklozenge$ is continuous;

(c) $a \blacklozenge 0 = a$ for all $a \in [0,1]$;

(d) $a \blacklozenge b \leq c \blacklozenge d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$:

Definition 1.3 A three tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ a continuous t-norm and M a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following condition, for all $x, y, z \in X$ and $t, s > 0$:

$(fm-1)$ $M(x, y, 0) = 0$

$(fm-2)$ $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,

$(fm-3)$ $M(x, y, t) = M(y, x, t)$,

$(fm-4)$ $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$,

$(fm-5)$ $M(x, y, .): [0, \infty) \longrightarrow [0,1]$ is left continuous,

$(fm-6)$ $\lim_{n\to\infty} M(x, y, t) = 1$.

Example1.1 Let (X, d) be a metric space , $(a * b) = \min(a, b)$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$, for all x, y in X and $t > 0$.

Then $M(x, y, *)$ is a fuzzy metric space often referred as standard fuzzy metric space induced by (X, d). **Definition 1.4.** A 5-tuple $(X, M, N, *, \blacklozenge)$ is said to be an intuitionistic fuzzy metric space (shortly IFM-Space) if X is an arbitrary set, $*$ is a continuous t-norm, $\blacklozenge$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$satisfying the following conditions:

$(IFm-1)$ $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;

$(IFm-2)$ $M(x, y, 0) = 0$ for all $x, y \in X$;

$(IFm-3)$ $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$,

$(IFm-4)$ $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

$(IFm-5)$ $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ for all $x, y, z \in X$ and $s, t > 0$,

$(IFm-6)$ $M(x, y, .): [0, \infty) \longrightarrow [0,1]$ is left continuous for all $x, y \in X$

(IFm – 7) $\lim_{n\to\infty} M(x,y,t) = 1$,
(IFm – 8) $N(x,y,0) = 1$ for all $x,y \in X$
(IFm – 9) $N(x,y,t) = 0$ for all $x,y \in X$ and $t > 0$ if and only if $x = y$,
(IFm – 10) $N(x,y,t) = N(y,x,t)$ for all $x,y \in X$ and $t > 0$,
(IFm – 11) $N(x,y,t) \blacklozenge N(y,z,s) \geq N(x,z,t+s)$ for all $x,y,z \in X$ and $s,t > 0$,
(IFm – 12) $M(x,y,.) : [0,\infty) \to [0,1]$ is right continuous for all $x,y \in X$ (IFm – 13) $\lim_{n\to\infty} N(x,y,t) = 0$ for all $x,y \in X$;
Then (M,N) is called an intuitionistic fuzzy metric on X . The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and degree of non nearness between x and y with respect to t, respectively.

Definition 1.5 Let $(X, M, N, *, \blacklozenge)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,
$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in\in X$ if, for all $t > 0$,
$\lim_{n\to\infty} M(x_n, x_n, t) = 1$, $\lim_{n\to\infty} N(x_n, x_n, t) = 0$

Lemma-1.1 Let $(X, M, N, *, \blacklozenge)$ be a sequence intuitionistic fuzzy metric space such that for all $x,y \in X$, $M(x,y,t) \to 1$ and $N(x,y,t) \to 0$ as $t \to \infty$. If there exist a constant $0 < k < 1$ such that, for all $t > 0$,
$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$ and $N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$
for all $x,y \in X$, then $x = y$.

Proof – If $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$ and $N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$ for all $t > 0$ and some constant $0 < k < 1$, then we have

$M(x,y,s) \geq M\left(x,y,\frac{s}{k}\right) \geq M\left(x,y,\frac{s}{k^2}\right) \geq \cdots \geq M\left(x,y,\frac{s}{k^n}\right)$, and

$N(x,y,s) \leq N\left(x,y,\frac{s}{k}\right) \leq N\left(x,y,\frac{s}{k^2}\right) \leq \cdots \leq N\left(x,y,\frac{s}{k^n}\right)$

for all $s > 0$ and $x,y \in X$. Letting $n \to \infty$, we have $M(x,y,s) = 1$ and $N(x,y,s = 0)$ and thus $x = y$.

Definition 1.6 An intuitionistic fuzzy metric space $(X, M, N, *, \blacklozenge)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 1.7 An intuitionistic fuzzy metric space $(X, M, N, *, \blacklozenge)$ is said to be compact if every sequence in X contains a convergent subsequence.

Definition 1.8 Let A and B be mappings from a intuitionistic fuzzy etric space $(X, M, N, *, \blacklozenge)$ into itself. The mappings A and B are said to be compatible if, for all $t > 0, M(ABx_n, BAx_n, t) \to 1$ and $N(ABx_n, BAx_n, t) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ as $n \to \infty$ for some $z \in X$.

Definition 1.9 Let A and B be mappings from a intuitionistic fuzzy metric space $(X, M, N, *, \blacklozenge)$ into itself. The mappings A and B are said to be compatible of type (α) if for all $t > 0$, $M(ABx_n, BAx_n, t) \to 1$ and $M(AAx_n, BAx_n, t) \to 1$ and $N(ABx_n, BAx_n, t) \to 0$ and $N(AAx_n, BAx_n, t) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ as $n \to \infty$ for some $z \in X$.

Definition 1.10 Let A and B be mappings from a intuitionistic fuzzy metric space $(X, M, N, *, \blacklozenge)$ into itself. The mappings A and B are said to be compatible of type (β) if for all $t > 0$, $M(AAx_n, BBx_n, t) \to 1$ and $N(AAx_n, BBx_n, t) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ as $n \to \infty$ for some $z \in X$.

1.3 Implicit relation

Let $I = [0,1]$, $*$ be a continuous t-norm and $\blacklozenge$ continuous t-co-norm and $F: I^6 \to R$ be a continuous function. Now, we consider the following conditions:
(F-1) F is non increasing in the fifth and sixth variables,
(F-2) it, for some constant $k \in (0,1)$, we have

(F_a) $F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1$

or

(F_b) $F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) \blacklozenge v\left(\frac{t}{2}\right)\right) \leq 0$

For any fixed $t > 0$ and any non decreasing functions $u, v: (0, \infty) \longrightarrow I$ with $0 < u(t), v(t) \leq 1$ then their exists $h \in (0,1)$ with $u(ht) \geq v(t) * u(t)$,

(F-3) if, for some constant $k \in (0,1)$, we have $F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$ and $F(u(kt), u(t), 0, 0, u(t), u(t)) \leq 0$ for any fixed $t > 0$ and any non decreasing function $u: (0, \infty) \longrightarrow I$, then $u(kt) \geq u(t)$ and $u(kt) \leq u(t)$.

Now, let F be the set of all real continuous functions $F: I^6 \longrightarrow R$ satisfying the conditions $(F-1) \sim (F-3)$.

Example 1.2 Let $F(u_1, \dots \dots, u_6) = \frac{u}{\min(u_2, \dots \dots, u_6)}$ and $a * b = \min(a, b)$ and $a \blacklozenge b = \max(a, b)$. Let

$t > 0, 0 < u(t), v(t) \leq 1, k \in (0, \frac{1}{2})$, where $u, v: (0, \infty) \longrightarrow I$ are non decreasing functions. Now suppose that

$F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1$, and $F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) \blacklozenge v\left(\frac{t}{2}\right)\right) \leq 0$ i.e.,

$F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) = \frac{u(kt)}{\min\{v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\}}$

$= \frac{u(kt)}{\min\{u\left(\frac{t}{2}\right), v\left(\frac{t}{2}\right)\}} \geq 1$, and

$F\left(u(kt), v(t), v(t), u(t), 0, u\left(\frac{t}{2}\right) \blacklozenge v\left(\frac{t}{2}\right)\right) = \frac{u(kt)}{\max\{v(t), u(t), 0, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\}}$

$= \frac{u(kt)}{\max\{u\left(\frac{t}{2}\right), v\left(\frac{t}{2}\right)\}} \leq 0$,

Thus $u(ht) \geq v(t) * u(t)$, *if* $h = 2k \in (0,1)$. *A similar argument works if* (F_b) *is assumed. Finally, suppose that* $t > 0$ *is fixed,* $u: (0, \infty) \longrightarrow I$ *is a non decreasing function and*

$(u(kt), u(t), 1, 1, u(t), u(t)) = \frac{u(kt)}{u(t)} \geq 1$, *and*

$(u(kt), u(t), 0, 0, u(t), u(t)) = \frac{u(kt)}{u(t)} \leq 0$,

for some $k \in (0,1)$. *Then we have* $u(kt) \geq u(t)$ *and* $u(kt) \leq u(t)$.

1.4 Main Result

Theorem 1.1 *Let* $(X, M, N, *, \blacklozenge)$ *be a complete intuitionistic fuzzy metric space with* $(a * b) = \min(a, b)$ *and* $(a \blacklozenge b) = \max(a, b)$ *for all* $a, b \in I$ *and A, B, S and T be mappings from X into itself satisfying the conditions:*

(1) $S(X) \subseteq B(X)$ *and* $T(X) \subseteq A(X)$,

(2) one on the mappings S,T,A,B is continuous,

(3) S and A as well as T and B are compatible of type (α),

(4) there exist $k \in (0,1)$ *such that*

$$F\begin{pmatrix} M(Sx, Ty, kt), M(Ax, By, t), M(Sx, Ax, t), M(Ty, By, t), \\ M(Sx, By, t), M(Ty, Ax, t) \end{pmatrix} \leq 1, \text{ and}$$

$$F\begin{pmatrix} N(Sx, Ty, kt), N(Ax, By, t), N(Sx, Ax, t), N(Ty, By, t), \\ N(Sx, By, t), N(Ty, Ax, t) \end{pmatrix} \leq 0$$

for all $x, y \in X$ *and* $t > 0$.

(5) for all $x, y \in X, M(x, y, t) \longrightarrow 1$ *and* $N(x, y, t) \longrightarrow 0$ *as* $t \longrightarrow \infty$,

Then A, B, S and T have a unique common fixed point in X.

Proof *– Let* x_0 *be an arbitrary point of X. From (1), we can construct a sequence* $\{y_n\}$ *in X as follows:*

$y_{2n+1} = Sx_{2n} = Bx_{2n+1}$ *and* $y_{2n+2} = Tx_{2n+1} = Bx_{2n+2}$, *for all* $n = 0,1,2 \dots$. *Then by (4), we have, for any* $t > 0$,

$$F\begin{pmatrix} M(Sx_{2n}, Tx_{2n+1}, kt), M(Ax_{2n}, Bx_{2n+1}, t), M(Sx_{2n}, Ax_{2n}, t), \\ M(Tx_{2n+1}, Bx_{2n+1}, kt), M(Sx_{2n}, Bx_{2n+1}, t), M(Tx_{2n+1}, Ax_{2n}, t) \end{pmatrix} \geq 1, \text{ and}$$

$$F\begin{pmatrix} N(Sx_{2n}, Tx_{2n+1}, kt), N(Ax_{2n}, Bx_{2n+1}, t), N(Sx_{2n}, Ax_{2n}, t), \\ N(Tx_{2n+1}, Bx_{2n+1}, kt), N(Sx_{2n}, Bx_{2n+1}, t), N(Tx_{2n+1}, Ax_{2n}, t) \end{pmatrix} \leq 0, \text{ and so}$$

$$F\begin{pmatrix} M(Sx_{2n}, Tx_{2n+1}, kt), M(Tx_{2n-1}, Sx_{2n}, t), M(Sx_{2n}, Tx_{2n-1}, t), \\ M(Tx_{2n+1}, Sx_{2n}, kt), 1, M\left(Tx_{2n+1}, Sx_{2n}, \frac{t}{2}\right) * M\left(Sx_{2n}, Tx_{2n-1}, \frac{t}{2}\right) \end{pmatrix} \geq 1, \text{ and}$$

$$F\begin{pmatrix} N(Sx_{2n}, Tx_{2n+1}, kt), N(Tx_{2n-1}, Sx_{2n}, t), N(Sx_{2n}, Tx_{2n-1}, t), \\ N(Tx_{2n+1}, Sx_{2n}, kt), 0, N\left(Tx_{2n+1}, Sx_{2n}, \frac{t}{2}\right) \blacklozenge N\left(Sx_{2n}, Tx_{2n-1}, \frac{t}{2}\right) \end{pmatrix} \leq 0$$

By (F-2) in implicit function we have

$M(Sx_{2n}, Tx_{2n+1}, ht) \geq M(Sx_{2n}, Tx_{2n-1}, t) * M(Sx_{2n}, Tx_{2n+1}, t)$, *and*

$N(Sx_{2n}, Tx_{2n+1}, ht) \leq N(Sx_{2n}, Tx_{2n-1}, t) \blacklozenge N(Sx_{2n}, Tx_{2n+1}, t)$, *and so*

$M(y_{2n+1}, y_{2n+2}, ht) \geq M(y_{2n+1}, y_{2n}, t) * M(y_{2n+1}, y_{2n+2}, t)$, and

$N(y_{2n+1}, y_{2n+2}, ht) \leq N(y_{2n+1}, y_{2n}, t) \blacklozenge N(y_{2n+1}, y_{2n+2}, t)$,

Which implies (note $(a * b) = min(a, b)$ and $(a \blacklozenge b) = max(a, b)$) that

$M(y_{2n+1}, y_{2n+2}, ht) \geq M(y_{2n+1}, y_{2n}, t) = M(y_{2n}, y_{2n+1}, t)$, and

$N(y_{2n+1}, y_{2n+2}, ht) \leq N(y_{2n+1}, y_{2n}, t) = N(y_{2n}, y_{2n+1}, t)$

Again by (F-2), we have

$M(y_{2n+1}, y_{2n}, ht) \geq M(y_{2n}, y_{2n-1}, t)$

Similar

$N(y_{2n+1}, y_{2n}, ht) \leq N(y_{2n}, y_{2n-1}, t)$

In general, we have, for all $m = 0,1,2,3, ...$ and $t > 0$,

$M(y_{m+1}, y_{m+2}, ht) \geq M(y_{m+1}, y_m, t) = M(y_m, y_{m+1}, t)$

and

$N(y_{m+1}, y_{m+2}, ht) \leq N(y_{m+1}, y_m, t) = N(y_m, y_{m+1}, t) \dots\dots\dots (1.1)$

To prove that $\{y_n\}$ is a Cauchy sequence, first, we prove that, for any $0 < \lambda < 1$ and $t > 0$.

$M(y_{n+1}, y_{n+m+1}, t) > 1 - \lambda$ and $M(y_{n+1}, y_{n+m+1}, t) < 0 - \lambda \dots\dots (1.2)$

for all $n \geq n_0$ and $m \epsilon N$. Here we use induction. From (1), we have

$M(y_{n+1}, y_{n+2}, t) \geq M\left(y_n, y_{n+1}, \frac{t}{h}\right) \geq \cdots \geq M\left(y_1, y_2, \frac{t}{h^n}\right) \longrightarrow 1$, and

$N(y_{n+1}, y_{n+2}, t) \geq N\left(y_n, y_{n+1}, \frac{t}{h}\right) \leq \cdots \leq N\left(y_1, y_2, \frac{t}{h^n}\right) \longrightarrow 0$

as $n \longrightarrow \infty$, i.e., for any $t > 0$ and $\lambda \epsilon (0,1)$, we can choose $n_0 \epsilon N$ such that

$M(y_{n+1}, y_{n+2}, t) > 1 - \lambda$ and $M(y_{n+1}, y_{n+2}, t) < 0 - \lambda \dots\dots (1.3)$

Thus (2) is true for $m = 1$. Suppose that (2) is true for $m \epsilon N$. Then we shall show that it is also true for $m + 1 \epsilon N$. Using the definition of intuitionistic fuzzy metric space, (1) and (2), we have

$M(y_{n+1}, y_{n+m+2}, t) \geq M\left(y_{n+1}, y_{n+m+1}, \frac{t}{2}\right) * M\left(y_{n+m+1}, y_{n+m+2}, \frac{t}{2}\right) \geq 1 - \lambda$, and

$N(y_{n+1}, y_{n+m+2}, t) \leq N\left(y_{n+1}, y_{n+m+1}, \frac{t}{2}\right) \blacklozenge N\left(y_{n+m+1}, y_{n+m+2}, \frac{t}{2}\right) \leq 0 - \lambda$.

Hence (2) is true for $m + 1 \epsilon N$. Thus $\{y_n\}$ is a Cauchy sequence in X. Since $(X, M, N, *, \blacklozenge)$ is complete, $\{y_n\}$ converge to a point $z \epsilon X$. Since $\{Ax_{2n+2}\}, \{Bx_{2n+2}\}, \{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ are subsequence of $\{y_n\}$, we have $Ax_{2n+2}, Bx_{2n+1}, Sx_{2n}, Tx_{2n+1} \longrightarrow z$ as $n \longrightarrow \infty$.

Now, suppose that A is continuous. Then the sequence $\{ASx_{2n}\}$ converges to Az as $n \longrightarrow \infty$. Notice that, for any $t > 0$,

$M(SAx_{2n}, Az, t) \geq M(SAx_{2n}, AAx_{2n}, \frac{t}{2}) * M(AAx_{2n}, Az, \frac{t}{2})$, and

$N(SAx_{2n}, Az, t) \leq N(SAx_{2n}, AAx_{2n}, \frac{t}{2}) \blacklozenge N(AAx_{2n}, Az, \frac{t}{2})$

Now, since A is continuous and S, A are compatible of type (α), letting $n \longrightarrow \infty$, we deduce that the sequence $\{SAx_{2n}\}$ converges to Az as $n \longrightarrow \infty$. Using (4), we have, for any $t > 0$,

$$F\begin{pmatrix} M(SAx_{2n}, Tx_{2n+1}, kt), M(AAx_{2n}, Bx_{2n+1}, t), M(SAx_{2n}, AAx_{2n}, t), \\ M(Tx_{2n+1}, Bx_{2n+1}, t), M(SAx_{2n}, Bx_{2n+1}, t), M(Tx_{2n+1}, AAx_{2n}, t) \end{pmatrix} \geq 1, \text{ and}$$

$$F\begin{pmatrix} N(SAx_{2n}, Tx_{2n+1}, kt), N(AAx_{2n}, Bx_{2n+1}, t), N(SAx_{2n}, AAx_{2n}, t), \\ N(Tx_{2n+1}, Bx_{2n+1}, t), N(SAx_{2n}, Bx_{2n+1}, t), N(Tx_{2n+1}, AAx_{2n}, t) \end{pmatrix} \leq 0$$

and then, by letting $n \longrightarrow \infty$, since F is continuous, we have

$F(M(Az, z, kt), M(Az, z, t), 1, 1, M(Az, z, t), M(Az, z, t)) \geq 1$, and

$F(N(Az, z, kt), N(Az, z, t), 0, 0, N(Az, z, t), N(Az, z, t)) \leq 0$,

Therefore, from (F-3), we have $M(Az, z, kt) \geq M(Az, z, t)$ and $N(Az, z, kt) \leq N(Az, z, t)$. From Lemma-2 , we have $Az = z$. Further more, by (4), we have

$$F\begin{pmatrix} M(Sz, Tx_{2n+1}, kt), M(Az, Bx_{2n+1}, t), M(Az, Sz, t), \\ M(Tx_{2n+1}, Bx_{2n+1}, t), M(Sz, Bx_{2n+1}, t), M(Tx_{2n+1}, Az, t) \end{pmatrix} \geq 1, \text{ and}$$

$$F\begin{pmatrix} N(Sz, Tx_{2n+1}, kt), N(Az, Bx_{2n+1}, t), N(Az, Sz, t), \\ N(Tx_{2n+1}, Bx_{2n+1}, t), N(Sz, Bx_{2n+1}, t), N(Tx_{2n+1}, Az, t) \end{pmatrix} \leq 0$$

and, letting $n \longrightarrow \infty$, we get

$F(M(Sz, z, kt), 1, M(Sz, z, t), 1, M(Sz, z, t), 1)) \geq 1$, and

$F(N(Sz, z, kt), 0, N(Sz, z, t), 0, N(Sz, z, t), 0)) \leq 0$

On the other hand, since

$M(Sz, z, kt) \geq M\left(Sz, z, \frac{t}{2}\right) = M\left(Sz, z, \frac{t}{2}\right) * 1$, and

$N(Sz, z, kt) \leq N\left(Sz, z, \frac{t}{2}\right) = N\left(Sz, z, \frac{t}{2}\right) \blacklozenge 0$, *and F is nonincreasing in the fifth variable, we have, for any* $t > 0$,

$F\left(M(Sz,z,kt),1,M(Sz,z,t),1,M\left(Sz,z,\frac{t}{2}\right)*1,1)\right)$

$\geq F(M(Sz,z,kt),1,M(Sz,z,t),1,M(Sz,z,t),1)) \geq 1$, *and*

$F\left(N(Sz,z,kt),0,N(Sz,z,t),0,N\left(Sz,z,\frac{t}{2}\right)\blacklozenge 0,0\right) \geq F(N(Sz,z,kt),0,N(Sz,z,t),0,N(Sz,z,t),0)) \leq 0$,

Which implies, by (F-2), that $Sz = z$. *This means that z is the range of S and, since* $S(X) \subseteq B(X)$, *there exists a point* $u \epsilon X$ *such that* $Bu = z$. *Using (4), we have successively*

$F(M(Sz,Tu,kt),M(Az,Bu,t),M(Sz,Az,t),M(Tu,Bu,t),M(Sz,Bu,t),M(Tu,Az,t))$

$= F(M(z,Tu,kt),1,1,M(z,Tu,t),1,M(z,Tu,t)) \geq 1$

and $F(N(Sz,Tu,kt),N(Az,Bu,t),N(Sz,Az,t),N(Tu,Bu,t),N(Sz,Bu,t),N(Tu,Az,t))$

$= F(N(z,Tu,kt),0,0,N(z,Tu,t),0,N(z,Tu,t)) \leq 0$

which implies, by (F-2), that $z = Tu$. *Since* $Bu = Tu = z$ *and B, T are compatible of type* (α), *we have* $TTu = BTu$ *so* $Tz = TTu = BTu = Bz$. *Therefore, from (4), we have, for any* $t > 0$,

$F(M(Sz,Tz,kt),M(Az,Bz,t),M(Sz,Az,t),M(Tz,Bz,t),M(Sz,Bz,t),M(Tz,Az,t))$

$= F(M(z,Tz,kt),M(z,Tz,t),1,1,M(z,Tz,t),M(z,Tz,t)) \geq 1$, *and*

$F(N(Sz,Tz,kt),N(Az,Bz,t),N(Sz,Az,t),N(Tz,Bz,t),N(Sz,Bz,t),N(Tz,Az,t))$

$= F(N(z,Tz,kt),N(z,Tz,t),0,0,N(z,Tz,t),N(z,Tz,t)) \leq 0$

Thus from (F-3), we have $M(z,Tz,kt) \geq M(z,Tz,t)$ *and* $N(z,Tz,kt) \leq N(z,Tz,t)$. *Again, from lemma-7.1 We have* $z = Tz = Bz$. *Consequently, z is a common fixed point of S, T, A and B. The same result holds if we assume that B is continuous instead of A.*

Now, we suppose that S is continuous. That the sequence $\{SAx_{2n}\}$ *converges to* Sz *as* $n \to \infty$. *Notice that, for any* $t > 0$,

$M(SAx_{2n},Sz,t) \geq M(ASx_{2n},SSx_{2n},\frac{t}{2}) * M(SSx_{2n},Sz,\frac{t}{2})$, *and*

$N(SAx_{2n},Sz,t) \leq N(ASx_{2n},SSx_{2n},\frac{t}{2})\blacklozenge N(SSx_{2n},Sz,\frac{t}{2})$,

Now, since S is continuous and S, A are compatible of type (α), *letting* $n \to \infty$, *we deduce that the sequence* $\{ASx_{2n}\}$ *converges to* Sz. *Using (4), we have, for any* $t > 0$,

$$F\begin{pmatrix} M(SSx_{2n},Tx_{2n+1},kt),M(ASx_{2n},Bx_{2n+1},t),M(SSx_{2n},ASx_{2n},t), \\ M(Tx_{2n+1},Bx_{2n+1},t),M(SSx_{2n},Bx_{2n+1},t),M(Tx_{2n+1},Ax_{2n},t) \end{pmatrix} \geq 1, \text{ and}$$

$$F\begin{pmatrix} N(SSx_{2n},Tx_{2n+1},kt),N(ASx_{2n},Bx_{2n+1},t),N(SSx_{2n},ASx_{2n},t), \\ N(Tx_{2n+1},Bx_{2n+1},t),N(SSx_{2n},Bx_{2n+1},t),N(Tx_{2n+1},Ax_{2n},t) \end{pmatrix} \leq 0$$

and than, by letting $n \to \infty$, *since F is continuous, we have*

$F(M(Sz,z,kt),M(Sz,z,t),1,1,M(Sz,z,t),M(Sz,z,t)) \geq 1$ *and*

$F(N(Sz,z,kt),N(Sz,z,t),0,0,N(Sz,z,t),N(Sz,z,t)) \leq 0$

Thus, from (F-3), we have $M(Sz,z,kt) \geq M(Sz,z,t)$ *and* $N(Sz,z,kt) \leq N(Sz,z,t)$. *Again, from lemma 7.1, we have* $Sz = z$. *This mean that z is the rang of S and, since* $S(X) \subseteq B(X)$, *there exists a point* $v \epsilon X$ *such that* $Bz = z$. *Using (4), we have, for any* $t > 0$,

$$F\begin{pmatrix} M(SSx_{2n},Tv,kt),M(ASx_{2n},Bv,t),M(SSx_{2n},ASx_{2n},t), \\ M(Tv,Bv,t),M(SSx_{2n},Bv,t),M(Tv,ASx_{2n},t) \end{pmatrix} \geq 1, \text{and}$$

$$F\begin{pmatrix} N(SSx_{2n},Tv,kt),N(ASx_{2n},Bv,t),N(SSx_{2n},ASx_{2n},t), \\ N(Tv,Bv,t),N(SSx_{2n},Bv,t),N(Tv,ASx_{2n},t) \end{pmatrix} \leq 0$$

Letting $n \to \infty$, *we get*

$F(M(z,Tv,kt),1,1,M(z,Tv,t),1,M(z,Tv,t)) \geq 1$, *and*

$F(N(z,Tv,kt),0,0,N(z,Tv,t),0,N(z,Tv,t)) \leq 0$,

which implies by (F-2), that $z = Tv$. *Since* $Bv = Tv = z$ *and B, T are compatible of type* (α), *we have* $TBv = BB$ *and so* $Tz = TBv = BBv = Bz$. *This, from (4) we have*

$$F\begin{pmatrix} M(Sx_{2n},Tz,kt),M(Ax_{2n},Bz,t),M(Sx_{2n},Ax_{2n},t), \\ M(Tz,Bz,t),M(Sx_{2n},Bz,t),M(Tz,Ax_{2n},t) \end{pmatrix} \geq 1, \text{ and}$$

$$F\begin{pmatrix} N(Sx_{2n},Tz,kt),N(Ax_{2n},Bz,t),N(Sx_{2n},Ax_{2n},t), \\ N(Tz,Bz,t),N(Sx_{2n},Bz,t),N(Tz,Ax_{2n},t) \end{pmatrix} \leq 0$$

Letting $n \to \infty$, *we get*

$F(M(z,Tz,kt),M(z,Tz,t),1,1,M(z,Tz,t),M(z,Tz,t)) \geq 1$ *and*

$F(N(z,Tz,kt),N(z,Tz,t),0,0,N(z,Tz,t),N(z,Tz,t)) \leq 0$.

Thus $z = Tz = Bz$. *This means that z is the range of T and, since* $T(X) \subseteq A(X)$, *there exists* $w \epsilon X$ *such that* $Aw = z$. *Thus, from (4), we have, for any* $t > 0$,

$$F\begin{Bmatrix} M(Sw,Tz,kt), M(Aw,Bz,t), M(Sw,Aw,t), \\ M(Tz,Bz,t), M(Sw,Bz,t), M(Tz,Aw,t) \end{Bmatrix} = F(M(Sw,z,kt), 1, M(Sw,z,t), 1, M(Sw,z,t), 1) \geq 1$$

and

$$F\begin{Bmatrix} N(Sw,Tz,kt), N(Aw,Bz,t), N(Sw,Aw,t), \\ N(Tz,Bz,t), N(Sw,Bz,t), N(Tz,Aw,t) \end{Bmatrix} = F(N(Sw,z,kt), 0, N(Sw,z,t), 0, N(Sw,z,t), 0) \leq 0$$

And, by $(F-2)$, we have $z = Sw = Aw$. Since $Sw = Aw = z$ and S, A are compatible of type (α), we have $z = Sz = SAw = AAw = Az$ and thus $z = Az$. Consequently, z is a common fixed point of S, T, A and B. the same result holds if we assume that T is continuous instead of S.

Finally, we show that the point z is unique common fixed point of S, T, A and B. Suppose that S, T, A and B have another common fixed point z_1. Then, by (4), we have, for any $t > 0$,

$$F\begin{pmatrix} M(Sz,Tz_1,kt), M(Az,Bz_1,t), M(Sz,Az,t), \\ M(Tz_1,Bz_1,t), M(Sz,Bz_1,t), M(Tz_1,Az,t) \end{pmatrix} = F(M(z,z_1,kt), M(z,z_1,t), 1, 1, M(z,z_1,t), M(z,z_1,t)) \geq 1, \text{ and}$$

$$F\begin{Bmatrix} N(Sz,Tz_1,kt), N(Az,Bz_1,t), N(Sz,Az,t), \\ N(Tz_1,Bz_1,t), N(Sz,Bz_1,t), N(Tz_1,Az,t) \end{Bmatrix} = F(N(z,z_1,kt), N(z,z_1,t), 0, 0, N(z,z_1,t), N(z,z_1,t)) \leq 0$$

Thus, (F-3), we have $M(z,z_1,kt) \geq M(z,z_1,t)$ and $N(z,z_1,kt) \leq N(z,z_1,t)$. From lemma-7.1, we have $z = z_1$. This completes the proof.

References

[1] Deng, Z.K.:Fuzzy pseudo metric spaces, Math. Anal. Appl., 86(1982), 84-75.
[2] Erceg, M.A.: Metric spaces in set theory, J. Math. Anal. Appl., 69(1979), 205-230.
[3] Karmosil, O. and Michalek, J.:Fuzzy metric and statistical metric spaces, Kybernetica, 11(1975),326-334.
[4] Keleva,O. and Seikkala,S.: On fuzzy metric spaces, Fuzzy Sets and Systems, 12(1984),225-229.
[5] Kutukcu, S.: Weak Compatibility and common coincidence points in intuitionistic Fuzzy metric spaces, Southeast Asian Bulletin of Mathematics, 32(2008), 1081- 1089.
[6] Zadeh, A.: Fuzzy sets, Inform control, 8(1965), 338-353.