# Approximation of Conjugate of Function Belonging to W($L_r$, ξ (t)) Class by (E, 2) (C, 1) Mean of Conjugate Fourier Series

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**Abstract:** ***In This paper a theorem on the degree of approximation of the conjugate of a function belonging to W($L_r$ ,ξ(t)) class by (E, 2) (C, 1) mean of Conjugate Fourier Series.***

**Keywords:** Degree of approximation, W($L_r$,ξ(t)) Class, (E, 2) (C, 1) means, Conjugate Fourier Series, summability method.

## 1. Introduction

A good amount of work to determine the degree of approximation of function belonging to the class. W($L_r$ ,ξ(t)) by Ceasaro, Norlund, Euler means has been done by several mathematician like Qureshi [ 8 ] ,Lal and Singh [ 4 ], Nigam [ 6 ], Dhakal [ 1]. In present work we determined the degree of approximation of the conjugate of function f belongs to, W($L_r$, ξ(t)) using ( E, 2) (C, 1) means Conjugate Fourier Series.

Let f be 2π periodic integrable over (- π, π) in the sense of Lebesgue then its Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(\cos nx + \sin nx) \qquad (1.1)$$

with partial sum $S_n(x)$. The conjugate Fourier series of (1.1) is given by

$$\sum_{n=1}^{\infty}(\sin nx - \cos nx) \qquad (1.2)$$

with partial sum $\overline{S_n}(x)$.

$$L_r \text{ —norm is defined by } \|f\|r = \left(\int_0^{2\pi}|f(x)|^r\,dx\right)^{\frac{1}{r}} \; r \geq 1 \qquad (1.3)$$

The degree of approximation $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = min\|t_n - f\|_r \qquad (1.4)$$

A function $f \in Lip\alpha$ if $f(x+t) - f(t) = o(|t|^{\propto})$ for

$$0 \leq \propto \leq 1 \qquad (1.5)$$

A function $f \in Lip(\alpha, r)$ for

$$0 \leq x \leq 2\pi, \left(\int_0^{2\pi}|f(x+t) - f(t)|^r\,dx\right)^{\frac{1}{r}} = o(|t|^{\propto}) \qquad (1.6)$$

For $0 \leq \propto \leq 1$ *and* $r \geq 1$.

Given a positive increasing function ξ(t) and an integer $r \geq 1, f(x) \in Lip(\xi(t), r)$

$$\left(\int_0^{2\pi}|f(x+t) - f(t)|^r\,dx\right)^{\frac{1}{r}} = o(\xi(t)) \qquad (1.7)$$

and that. $fs(x) \in W(L_r\,\xi(t))$ if

$$\left(\int_0^{2\pi}|f(x+t) - f(t)|^r \sin^{\beta r} x dx\right)^{\frac{1}{r}} = o(\xi(t)) \quad ,\beta \geq o, r \geq 1 \qquad (1.8)$$

If $\beta = 0$ then $W(L_r\,\xi(t))$ reduces to the class $Lip(\xi(t), r)$ if $\xi(t) =)t^{\alpha}$ then $Lip(\xi(t), r)$ reduces to the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ reduces to class $Lip\alpha$. We observe that $Lip \propto \subseteq Lip(\propto, r) \subseteq Lip(\xi(t), r) \subseteq W(L_r\,\xi(t))$. for $0 < \propto \leq 1, r \geq 1$.

Let $\sum_0^{\infty} u_n$ be a given infinite series with the sequence of its $n^{th}$partial sum $\{s_n\}$. The (C,1) transform is defined as the $n^{th}$partialsum of (C,1) summability and is given by

$$t_n = \frac{S0+S1+S2+\cdots Sn}{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s \text{ as } n \to \infty. \qquad (1.9)$$

$$\text{If } (E,q) = E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} s_k \to s \text{ as } n \to \infty. \qquad (1.10)$$

Then the infinite series $\sum_0^{\infty} u_n$ is said to be summable (E,q) to a definite numbers.

## 2. Main Results

Theorem – If $\bar{f}$ conjugate to a $2\pi$- periodic function belongs to $W(L_r\,\xi(t))$ class then its degree of approximation by (E,2) (C,1) means of conjugate Fourier series is given by

$$\left\|\overline{(E_2C)}_n^1 - \bar{f}\right\|_r = o\left[(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right] \qquad (2.1)$$

Provided ξ(t) satisfies the following condition $\left\{\frac{\xi(t)}{t}\right\}$ be a non-increasing sequence

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^r \sin^{\beta r} t\,dt\right\}^{\frac{1}{r}} = o\left(\frac{1}{n}\right) \qquad (2.2)$$

$$and \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^{r} dt\right\}^{\frac{1}{r}} = o\{(n+1)^{\delta}\} \qquad (2.3)$$

Uniformaly in x. Where $\delta$ is an arbitrary number such that $s(1-\delta)-1>0, \frac{1}{r}+\frac{1}{s}=1$ and $\overline{(E_2C)}_n^1$ as defined (E,2) (C,1) means of conjugate Fourier series and $\bar{f}(x) = -\frac{1}{2\pi}\int_0^{2\pi}\psi(t)\cot\left(\frac{t}{2}\right)dt.$

For the proof of our theorem following lemmas are required.

**Lemma1**. – $|\overline{N_n}(t)| = o\left(\frac{1}{t}\right)$ for $0 \le t \le \frac{1}{n+1}$.

**Proof**–For $0 \le t \le \frac{1}{n+1}, \sin\left(\frac{t}{2}\right) \ge o\left(\frac{t}{\pi}\right)$ and $|\cos nt| \le 1$.

$$|\overline{N_n}(t)| = \left|\frac{1}{3^n\pi}\sum_{k=0}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}\sum_{v=0}^{k}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|$$

$$\le \frac{1}{3^n\pi}\sum_{k=0}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}\sum_{v=0}^{k}\left|\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| \le$$

$$\frac{1}{3^n\pi}\sum_{k=0}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}\sum_{v=0}^{k}\frac{\left|\cos\left(v+\frac{1}{2}\right)t\right|}{\left|\sin\frac{t}{2}\right|}$$

$$\le \frac{1}{3^n t}\sum_{k=0}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}$$

$$= o\left(\frac{1}{t}\right).$$

**Lemma 2.** – For $0 \le a \le b \le \infty, 0 \le t \le \pi$ and any n we have $|\overline{N_n}(t)| = o\left[\frac{1}{t}\right]$.

Proof -

$$|\overline{N_n}(t)| = \left|\frac{1}{3^n\pi}\sum_{k=0}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}\sum_{v=0}^{k}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| \le$$

$$\frac{1}{3^n t}\left|\sum_{k=0}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}Re\left\{\sum_{v=0}^{k}e^{i\left(v+\frac{1}{2}\right)t}\right\}\right| \le$$

$$\frac{1}{3^n t}\left|\sum_{k=0}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}Re\left\{\sum_{v=0}^{k}e^{iv}\right\}\right|\left|e^{\frac{it}{2}}\right| \le$$

$$\frac{1}{3^n t}\left|\sum_{k=0}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}Re\left\{\sum_{v=0}^{k}e^{iv}\right\}\right| \le$$

$$\frac{1}{3^n t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}Re\left\{\sum_{v=0}^{k}e^{iv}\right\}\right| + \frac{1}{3^n t}\left|\sum_{k=\tau}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}Re\left\{\sum_{v=0}^{k}e^{iv}\right\}\right| .. (2.5)$$

Now consider first term of (2.5) we get

$$\frac{1}{3^n t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}Re\left\{\sum_{v=0}^{k}e^{iv}\right\}\right| \le \frac{1}{3^n t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}Re\left\{\sum_{v=0}^{k}1\right\}\right|\left|e^{ivt}\right|$$

$$\le \frac{1}{3^n t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}2^{n-k-1}\right| \ldots\ldots\ldots\ldots\ldots(2.6)$$

Again second terms of (2.5) we get

$$\frac{1}{3^n t}\left|\sum_{k=\tau}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}Re\left\{\sum_{v=0}^{k}e^{iv}\right\}\right| \le \frac{1}{3^n t}\sum_{k=\tau}^{n}\binom{n}{k}2^{n-k-1}\frac{1}{k+1}\max_{0\le m\le k}\left|\sum_{v=0}^{m}e^{ivt}\right|$$

$$\le \frac{1}{3^n t}\sum_{k=\tau}^{n}\binom{n}{k}2^{n-k-1}.$$

From (2.5) (2.6) and (2.7 we get

$$|\overline{N_n}(t)| \le \frac{1}{3^n t}\sum_{k=0}^{\tau-1}\binom{n}{k}2^{n-k-1} + \frac{1}{3^n t}\sum_{k=\tau}^{n}\binom{n}{k}2^{n-k-1}$$

$= o\left[\frac{1}{t}\right]$. $\tau = \frac{1}{t}$. Where $\tau$ denotes the greatest integer not greater than $\frac{1}{t}$.

Proof – Let $\bar{s_n}(x)$ be partial sum of conjugate Fourier series then we have

$$\bar{s_n}(x) - \bar{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}dt$$

The (C,1) transform of $\bar{s_n}(x)$ is given by $\overline{C_1^n} - \bar{f}(x) = \frac{1}{2\pi(n+1)}\int_0^{\pi}\psi(t)\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}dt$

Now the (E,2) (C,1) transform of $\bar{s_n}(x)$ by $\overline{(E_2C)}_n^1$ we can write

$$\overline{(E_2C)}_n^1 - \bar{f}(x)$$

$$= \frac{1}{3^n\pi}\sum_{k=0}^{n}\binom{n}{k}2^{n-k-1}\int_0^{\pi}\frac{\psi(t)}{\sin\frac{t}{2}}\frac{1}{k+1}\left\{\sum_{v=0}^{k}\cos\left(v+\frac{1}{2}\right)t\right\}dt$$

$$= \int_0^{\pi}\psi(t)\overline{N_n}(t)\,dt = \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\psi(t)\overline{N_n}(t)\,dt$$

$$= I_1 + I_2 \text{ (Say)}$$

Applying Holder inequality and $\psi(t)\epsilon\, w(L_r, \xi(t))$ and from (2.2), lemma 1 and second mean value theorem for integral we have

$|I_1|$

$$\le \left[\int_0^{\frac{1}{n+1}}\left\{\frac{t|\psi(t)|\sin^{\beta}t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)|\overline{N_n}(t)|}{t\sin^{\beta}t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{t^{2+\beta}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\xi}^{\frac{1}{n+1}}\frac{dt}{t^{(2+\beta)s}}\right]^{\frac{1}{s}}$$

$$= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-(2+\beta)s+1}}{-(2+\beta)s+1}\right\}_{\varepsilon}^{\frac{1}{n+1}}\right]^{\frac{1}{s}}$$

$$= O\left[\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)(n+1)^{2+\beta-\frac{1}{s}}\right]$$

$$= O\left[(n+1)^{1+\beta-\frac{1}{s}}\xi\left(\frac{1}{n+1}\right)\right]$$

$$= O\left[(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right]$$

Since $\frac{1}{r}+\frac{1}{s} = 1, 1 \le r \le \infty$ ...........(2.9)

Applying holder inequality $|\sin t| < 1, \sin t \ge \left(\frac{2t}{\pi}\right)$ from (2.3) and lemma 2 second mean value theorem for integral, we have

$|I_2|$

$$\le \left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\psi(t)|\sin^{\beta}t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)|\overline{N_n}(t)|}{t^{-\delta}\sin^{\beta}t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\{(n+1)^{\delta}\}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{1-\delta+\beta}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\{(n+1)^{\delta}\}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi(1/y)}{y^{\delta-1-\beta}}\right\}^s\frac{dy}{y^2}\right]^{\frac{1}{s}}, \text{ Putting } t = 1/y$$

$$= O\left\{(+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{(n+1)^{s(1+\beta-\delta)-1}-\pi^{s(\delta-1-\beta)+1}}{s(1+\beta-\delta)-1}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1+\beta-\delta)-\frac{1}{s}}\right]$$

$$= O\left\{(n+1)^{\beta+1-\frac{1}{s}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

$$O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}\ldots\ldots\ldots (2.10)$$

From (2.8) (2.9) and (2.10) we have $\left|\,(\overline{E_2C})_n^1 - \bar{f}\right| = O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}$

$$\|\,(\overline{E_2C})_n^1 - \bar{f}\|r = \left\{\int_0^{2\pi}\left|\,(\overline{E_2C})_n^1 - \bar{f}\right|^r dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof.

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