Smooth Graceful Graphs And Its Application To Construct Graceful Graphs

V J Kaneria[1], M M Jariya[2]

[1]Department of Mathematics, Saurashtra University, Rajkot, India
[2]V. V. P. Engineering College, Rajkot, India

Abstract: ***In this paper we define smooth graceful labeling and we prove that cycle C_n, complete bipartite graph $K_{2,n}$ and path P_n are smooth graceful graphs. Using this smooth graceful labeling we prove that a graph obtained by joining C_m^+ ($m \equiv 2 \pmod 4$) and C_n ($n \equiv 0 \pmod 4$) with a path of arbitrary length is graceful. We also prove a graph obtained by joining C_m ($m \equiv 0 \pmod 4$) and W_n with a path of arbitrary length is graceful.***

Keywords: Graceful labeling, smooth graceful labeling, cycle, path, wheel and complete bipartite graph.

AMS subject classification number : 05C78.

1. Introduction

Let $G=(V,E)$ be a simple, undirected graph of size (p,q) i.e. $|V|=p$, $|E|=q$. For all standard terminology and notations we follow Harary (Harary 1972). We will give brief summery of definitions which are used in this paper.

Definition $-1.1:$ If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

Most interesting graph labeling problems have three important ingrediants:

- A set of numbers from which vertex labels are chosen.
- A rule that assigns a value to each edge.
- A condition that these values must satisfies.

Definition -1.2 : A function f is called graceful labeling of a graph $G=(V,E)$ if $f:V\rightarrow\{0,1,\dots,q\}$ is injective and the induce function $f^{å}:E\rightarrow\{1,2,\dots,q\}$ defined as $f^{å}(e)$ = $|f(u)-f(v)|$ is bijective for every edge $e=(u,v)\in E$. A graph G is called *graceful graph* if it admits a graceful labeling.

The graceful labeling was introduced by A. Rosa (Rosa, 1967, p. 349 − 355). He proved that cycle C_n is graceful, when $n\equiv 0,3 \pmod 4$. Bloom and Golumb (Bloom and Golomb, 1978, p. 53 − 65) proved that the complete graph K_n is graceful, when $n\le 4$. Hoede and Kuiper (Hoede and Kuiper, 1987, 311), Frucht (Frucht, 1979, 219 − 229) proved that wheels $W_n=C_n+K_1$ are graceful. For detail survey of graph labeling one can refer Gallian (Gallian, 2013).

In this paper we introduce smooth graceful labeling and using this we prove that a graph obtained by joining C_m^+ ($m\equiv 2 \pmod 4$) and C_n ($n\equiv 0 \pmod 4$) with a path is graceful. We also prove a graph obtained by joining C_m ($m\equiv 0 \pmod 4$) and W_n with a path is graceful.

Definition -1.3 : A bipartite graceful graph G with graceful labeling f is said to be *smooth graceful graph* if it admits an injective map $g:V\rightarrow\{0,1,\dots,\lfloor\frac{q-1}{2}\rfloor,\lfloor\frac{q+1}{2}\rfloor+l,\lfloor\frac{q+3}{2}\rfloor+l, \dots,q+l\}$ such that its induce edge labeling map $g^{å}:E\rightarrow\{1+l,2+l,\dots,q+l\}$ defined as $g^{å}(e)=|g(u)-g(v)|$, for every edge $e=(u,v)\in E$, for any $l\in N$ is a bijection.

Smooth graceful graph will help to produce new disconnected as well as connected graceful graphs.

2. Main Results

Theorem -2.1 : A cycle C_n, ($n\equiv 0 \pmod 4$) is a smooth graceful graph.

Proof : Let $v_1,v_2,\dots,v_n$ be vertices of C_n. We know that C_n ($n\equiv 0 \pmod 4$) is a bipartite graph and $f:V(C_n)\rightarrow\{0,1,\dots,n\}$ defined by

$f(v_i) \;=\; \frac{i-1}{2}, \;\forall\; i = 1,3,\ldots,n-1.$

$= q-(\frac{i-2}{2}), \;\forall\; i = 2,4,\ldots,\frac{n}{2}$

$= q-(\frac{i}{2}), \;\forall\; i = \frac{n}{2}+2, \frac{n}{2}+4,\ldots,n$ is graceful labeling for C_n, when $n \equiv 0 \pmod 4$.

Now we define $g: V(C_n) \to \{0,1,\ldots,\lfloor\frac{n-1}{2}\rfloor, \lfloor\frac{n+1}{2}\rfloor+l, \lfloor\frac{n+3}{2}\rfloor+l,\ldots,n+l\}$ such that its induce edge labeling map $g^{\mathring{a}}: E(C_n) \to \{1+l, 2+l,\ldots,q+l\}$ defined by

$g(u) = f(u), \;\forall\; u \in \{v_1, v_3,\ldots,v_{n-1}\}$

$= f(u)+l, \;\forall\; u \in \{v_2, v_4,\ldots,v_n\}$ and

$g^{\mathring{a}}(e) = |g(u)-g(v)|$, for every edge $e = (u,v) \in E$.

Since for any $k \in \{1,2,\ldots,n\}$, $e_k = (v_k, v_{k+1}) \in E$, by taking $v_{n+1} = v_1$,

$g^{\mathring{a}}(e_k) = |g(v_k - g(v_{k+1})|$

$= |f(v_k - f(v_{k+1})|+l$

$= f^{\mathring{a}}(e_k)+l$, so $g^{\mathring{a}}(E) = \{1+l, 2+l,\ldots,n+l\}$.

Therefore $g^{\mathring{a}}$ is a bijective map. Hence C_n, ($n \equiv 0 \pmod 4$) is a smooth graceful graph.

Theorem -2.2 **:** A complete bipartite graph $K_{2,n}$ is a smooth graceful graph.

Proof : Let $u_1, u_2, v_1, v_2,\ldots,v_n$ be vertices of $K_{2,n}$.

We know that $f: V(K_{2,n}) \to \{0,1,\ldots,2n\}$ defined by

$f(v_i) \;=\; i-1, \;\forall\; i = 1,2,\ldots,n.$

$f(u_i) \;=\; in, \;\forall\; i = 1,2$ is a graceful labeling $K_{2,n}$.

Now we define $g: V(K_{2,n}) \to \{0,1,\ldots,n-1, n+l, n+1+l,\ldots,n+l\}$ and its induce edge labeling map $g^{\mathring{a}}: E(K_{2,n}) \to \{1+l, 2+l,\ldots,n+l\}$ defined by

$g(u) = f(u)+l, \;\forall\; u \in \{u_1, u_2\}$

$= f(v), \;\forall\; v \in \{v_1, v_2,\ldots,v_n\}$ and $g^{\mathring{a}}(e) = |g(u)-g(v)|, \;\forall\; e = (u,v) \in E$.

Since for any $e = (u_i, v_j) \in E$ $(i = 1,2, 1 \le j \le n)$,

$g^{\mathring{a}}(e) = |g(u_i) - g(v_j)|$

$= f(u_i) - f(v_j)|+l$

$= f^{\mathring{a}}(e)+l.$

So $g^{\mathring{a}}(E) = \{1+l, 2+l,\ldots,2n+l\}$. Therefore $g^{\mathring{a}}$ is a bijection.

Hence $K_{2,n}$ is a smooth graceful graph.

Theorem -2.3 **:** A path P_n of length $n-1$ is a smooth graceful graph, $\forall\; n \in N$.

Proof : Let $v_1, v_2,\ldots,v_n$ be vertices of P_n. We know that P_n is a bipartite and graceful graph with graceful labeling $f: V(P_n) \to \{0,1,\ldots,n-1\}$ defined by

$f(v_i) \;=\; \frac{i-2}{2}$, if $i \equiv 0 \pmod 2$

$= n-(\frac{i+1}{2})$, if $i \equiv 1 \pmod 2$, $\forall\; i = 1,2,\ldots,n.$

Now we define $g: V(P_n) \to \{0,1,\ldots,\lfloor\frac{n-2}{2}\rfloor, \lfloor\frac{n}{2}\rfloor+l, \lfloor\frac{n+2}{2}\rfloor+l,\ldots,n-1+l\}$ and its induce edge labeling map $g^{\mathring{a}}: E(P_n) \to \{1+l, 2+l,\ldots,n-1+l\}$ defined by

$g(v) \;=\; f(v)+l$, if $i \equiv 1 \pmod 2$

$= f(v)$, if $i \equiv 0 \pmod 2$, $\forall\; i = 1,2,\ldots,n$

and $g^{\mathring{a}}(e) = |g(u)-g(v)|$, for every $e = (u,v) \in E$.

Since for any $e = (v_i, v_{i+1}) \in E$ $(1 \le i \le n-1)$,

$g^{\mathring{a}}(e) = |g(v_i) - g(v_{i+1})|$

$= |f(v_i) - f(v_{i+1})|+l$

$= f^{\mathring{a}}(e)+l$, we shall have $g^{\mathring{a}}(E) = \{1+l, 2+l,\ldots,n-1+l\}$.

Therefore $g^{\mathring{a}}$ is a bijection and so P_n of length $n-1$ is a smooth graceful, $\forall\; n \in N$.

Theorem -2.4**:** Let C_m^+ ($m \equiv 2 \pmod 4$) be a cycle with twin chords, they form a triangle with an edge of the cycle C_m. Then the graph obtained by joining C_m^+ and a cycle C_n ($n \equiv 0 \pmod 4$) with a path P_{t+1} of arbitrary length t is graceful.

Proof : Let G be a graph obtained by joining C_m^+ and C_n with P_{t+1}, where C_m^+ is a cycle with twin chords and they form a triangle with an edge of the cycle C_m, C_n is a cycle on n vertices and P_{t+1} is a path on $t+1$ vertices. Let $u_1, u_2, \ldots, u_m$ be vertices of C_m^+ with twin chords $(u_2, u_m), (u_2, u_{m-1})$ $v_1 = u_{m-2}, v_2, \ldots, v_{t+1}$ be vertices of the path P_{t+1} of t length and $w_1 = v_{t+1}, w_2, \ldots, w_n$ be vertices of the cycle C_n.

We define labeling function $f: V(G) \to \{0, 1, \ldots, q\}$, where $q = m + t + n + 2$ as follows:

$f(u_1) = q - 1$, $f(u_m) = q$, $f(u_{m-1}) = q - 2$,

$f(u_j) = \frac{j-2}{2}$, $\forall\ j = 2, 4, \ldots, \frac{m}{2} - 1$

$= \frac{j}{2}$, $\forall\ j = \frac{m}{2} + 1, \frac{m}{2} + 3, \ldots, m - 2$

$= q - (\frac{j+3}{2})$, $\forall\ j = 3, 5 \ldots, m - 3$;

$f(v_i) = q - (\frac{m+i}{2})$, when $i \equiv 0 \pmod 2$

$= \frac{m+i-3}{2}$, when $i \equiv 1 \pmod 2$, $\forall\ i = 2, 3, \ldots, t+1$;

$f(w_k) = \frac{m+t+k-3}{2}$, $\forall\ k = 3, 5, \ldots, n-1$ and t is even or

$\forall\ k = 2, 4, \ldots, \frac{n}{2}$ and t is odd

$= q - (\frac{m+t+k}{2})$, $\forall\ k = 3, 5, \ldots, n-1$ and t is odd or

$\forall\ k = 2, 4, \ldots, \frac{n}{2}$ and t is even

$= q - (\frac{m+t+k+2}{2})$, $\forall\ k = \frac{n}{2} + 2, \frac{n}{2} + 4, \ldots, n$ and t is even

$= \frac{m+t+k-1}{2}$, $\forall\ k = \frac{n}{2} + 2, \frac{n}{2} + 4, \ldots, n$ and t is odd.

Above labeling pattern give rise graceful labeling to the given graph G.

Illustration $- 2.5$ **:** A cycle C_{10}^+ with twin chords, C_{12} joining by a path P_9 of 8 length with its graceful labeling shown in *figure* -1.

Theorem $- 2.6$ **:** A graph obtained by joining C_m ($m \equiv 0 \pmod 4$) and a wheel $W_n = C_n + K_1$ with a path of arbitrary length t is graceful.

Proof : Let G be a graph obtained by joining C_m and W_n with P_{t+1} a path of length t on $t+1$ vertices. Let $u_1, u_2, \ldots, u_m$ be vertices of C_m, $v_1 = u_m, v_2, \ldots, v_{t+1}$ be vertices of path P_{t+1} of t length and $w_0, w_1, \ldots, w_n = v_{t+1}$ be vertices of the wheel W_n in which w_0 is adjacent with each w_i ($1 \le i \le n$).

We know that wheel W_n is a graceful graph with labeling function f_1 defined as follows:

$f_1(w_0) = 0$, $f_1(w_n) = 2n$

$f_1(w_i) = 2n - i - 1$, if i is even ($2 \le i \le n-2$) and n is even

$= 2$, if $i = n - 1$ and n is even

$= i$, if i is odd ($1 \le i \le n-3$) and n is even

$= 2i$, if $i = 1$ and n is odd

$= n + i$, if i is even ($2 \le i \le n-1$) and n is odd

$= n + 1 - i$, if i is odd ($3 \le i \le n-2$) and n is odd.

We define function $f: V(G) \to \{0, 1, \ldots, q\}$, where $q = m + t + 2n$ as follows:

$f(u_j) = q - (\frac{j-1}{2})$, $\forall\ j = 1, 3, \ldots, m-1$.

$= \frac{j-2}{2}$, $\forall\ j = 2, 4, \ldots, \frac{m}{2}$

$= \frac{j}{2}$, $\forall\ j = \frac{m}{2} + 2, \frac{m}{2} + 4, \ldots, m$;

$f(v_i) = q + 1 - (\frac{m+i}{2})$, when $i \equiv 0 \pmod 2$

$= \frac{m+i-1}{2}$, when $i \equiv 1 \pmod 2$, $\forall\ i = 2, 3, \ldots, t$;

$f(w_k) = 2n + (\frac{m+t}{2}) - f_1(w_k)$, when t is even

$= f_1(w_k) + \frac{m+t+1}{2}$, when t is odd, $\forall\ k = 0, 1, \ldots, n$.

Above labeling pattern give rise graceful labeling to the given graph G.

Illustration -2.7 : A cycle C_{12}, a wheel W_7 joining by a path P_6 of 5 length with its graceful labeling shown in *figure* -2.

3. Concluding Remarks

We have introduced a new graph labeling is called smooth graceful labeling. We have proved that C_n $n \equiv 0$ (mod 4), $K_{2,n}$, P_n are smooth graceful graphs. Using these we have got graceful labeling for two new families of graphs. Present work contribute five new results. The labeling pattern is demonstrated by illustrations.

References

[1] G. S. Bloom and S. W. Golomb, Numbered complete graphs, usual rules and assorted applications, in *Theory and applications of Graphs, Lecture Notes in Math.*, 642, Springer $-$ Verlag, New York (1978) $53-65$.

[2] J. A. Gallian, *The Electronics Journal of Combinatorics*, 16, $\#$DS6(2013).

[3] R. Frucht, Graceful numbering of wheels and related graphs, *Ann. N. Y. Acad. Sci.*, 319 (1979) $219-229$.

[4] F. Harary, Graph theory *Addition Wesley, Massachusetts*, 1972.

[5] C. Hoede and H. Kuiper, All wheels are graceful, *Util. Math.*, 14 (1987) 311.

[6] A. Rosa, On certain valuation of graph, *Theory of Graphs (Rome, July 1966)*, Goden and Breach, N. Y. and Paris, 1967, $349-355$.

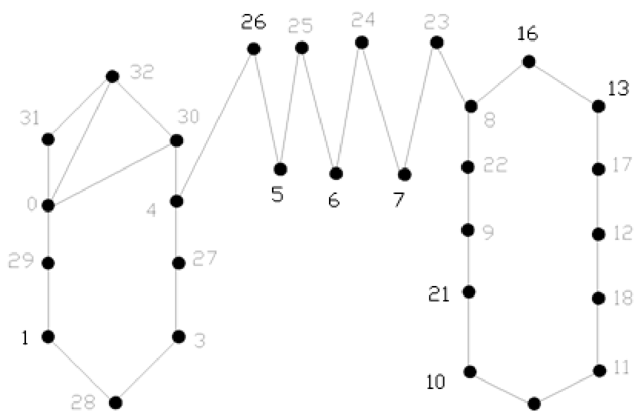

Figure 1: A cycle C_{10}^{+} with twin chords, C_{12} joining by a path P_9 of 8 length and its graceful labeling

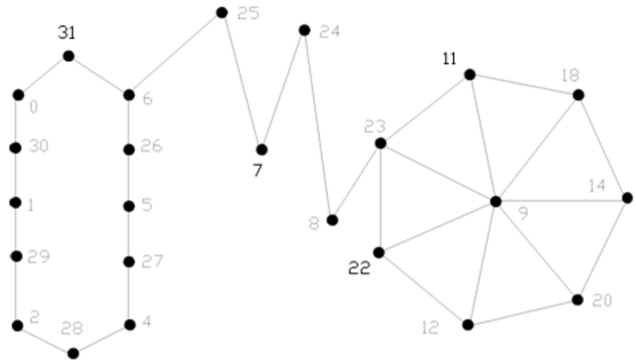

Figure 2: A cycle C_{12}, a wheel W_7 joining by a path P_6 of 5 length with its graceful labeling