Diffuseness Parameters of Woods–Saxon Potential for Heavy-Ion Systems through Large-Angle Quasi-Elastic Scattering

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Abstract: In this paper, analyses on the nuclear potential for heavy ion systems, namely \(^{48}\text{Ti},\ 54\text{Cr},\ \text{and} \ 64\text{Ni} + 208\text{Pb}\) systems, have been performed through large-angle quasi-elastic scattering at sub-barrier energies. At energies around the Coulomb barrier height, it has been well known that the effect of channel couplings, that is the coupling between the relative motion of the colliding nuclei and their intrinsic motions as well as transfer processes, plays an important role. Therefore, a coupled-channels procedure must be applied to take account of this effect. A modified version of a computer code ccfull has been employed in order to perform these complex calculations. The nuclear potential is assumed to have a Woods-Saxon form, which is characterized by the surface diffuseness parameter, the potential depth, and the radius parameter. In order to find the best fitted value of the diffuseness parameter in comparison with the experimental data, the chi square method \(\chi^2\) is used. The best fitted value of the diffuseness parameter for studying systems obtained through a coupled-channel calculation with inert Target and vibrational Projectile. The calculated ratio of the quasi-elastic to the Rutherford cross sections for \(^{48}\text{Ti},\ 54\text{Cr} \ \text{and} \ 64\text{Ni} + 208\text{Pb}\) systems give a good agreement using \(a = 0.44\ \text{fm},\ 0.67\ \text{fm}\) and \(a = 0.67\ \text{fm}\), respectively.

Keywords: Heavy-ion fusion reactions, quasi-elastic scattering, Coupled-channels calculations, sub-barrier energies.

1. Introduction

The knowledge of the potential between two colliding nuclei is of fundamental importance in order to describe nucleus-nucleus collisions. The nucleus-nucleus potential is the sum of a short range attractive nuclear potential \(V_r(r)\) and a long range repulsive Coulomb potential \(V_C(r)\). The Coulomb potential is well understood. This has been demonstrated by the accurate description of the Coulomb or Rutherford scattering, the scattering where only the long range Coulomb potential acts.

The nuclear potential can be studied through fusion or quasi-elastic scattering experimental data. Quasi-elastic scattering is the sum of elastic scattering, inelastic scattering and transfer reaction. Thus, quasi-elastic scattering and fusion are complementary to each other due to flux conservation. At zero impact parameter (i.e. head-on collision), quasi-elastic scattering is related to the reflection probability by the potential barrier, while fusion is related to the penetration probability \([1,2]\). The diffuseness parameter determines the characteristic at the surface region of the nuclear potential. Nuclear potential of the Woods-Saxon form, which is described by the potential depth \(V_0\), the radius parameter \(r_0\), and the diffuseness parameter \(a\), is widely used in the analyses of nuclear collisions.

In this study, we assume that the nuclear potential has a Woods-Saxon form. A diffuseness parameter of around 0.63 fm is widely accepted [3]. This has been supported by recent studies such as by Gasques et al. [4] and Evers et al. [5], where both studies performed analyses on the diffuseness parameter using the experimental data of large-angle quasi-elastic scattering. However, relatively higher diffuseness parameters are required in order to fit fusion data, as shown by Newton et al. [6] for example. The cause of the discrepancy is still not well understood. The aim of the present work is to analysis diffuseness parameters of Woods–Saxon potential for heavy-ion systems through large-angle quasi-elastic scattering at sub-barrier. The chi square method \(\chi^2\) is used to find the best fitted value of the diffuseness parameter in comparison with the experimental data.

2. Theory

2.1 The Nucleus-Nucleus Potential

The nucleus-nucleus potential, which is the sum of a short range attractive nuclear potential \(V_N\) and a long range repulsive Coulomb potential \(V_C\). The Coulomb potential between two spherical nuclei with uniform charge density distributions and when they do not overlap is given by [7]

\[
V_C(r) = \frac{Z_p Z_T e^2}{r} \quad (1)
\]

where \(Z_p, Z_T\), \(r\), and \(e\) are the atomic number of the projectile, the atomic number of the target, the distance between the centers of the colliding nuclei, and the elementary charge (Gaussian units), respectively. When the nuclei overlap, the Coulomb potential is given by [7]

\[
V_C(r) = \frac{Z_p Z_T e^2}{2r_C} \left[ 3 - \left(\frac{r}{r_C}\right)^2 \right], \quad (2)
\]
where $R_C$ is the radius of the equivalent sphere of the target and projectile. For the nuclear potential, the Woods-Saxon form is widely used, and is given by [8]

$$V_C(r) = -\frac{V_0}{1 + \exp\left[\frac{r - R_0}{a}\right]}$$  \hspace{1cm} (3)$$

where $V_0$ is the potential depth, $a$ is the surface diffuseness parameter, and $R_0 = r_0\left(A_T^{1/3} + A_P^{1/3}\right)$, where $r_0$ is the radius parameter, while $A_T$ and $A_P$ are the mass numbers of the target and the projectile, respectively.

### 2.2 Coupled-Channels Equation with Full Angular Momentum

Let us consider a collision between two nuclei in the presence of the coupling between the relative motion of the center of mass of the colliding nuclei, $\vec{r} = (r, \vec{p})$ and the nuclear intrinsic motion $\xi$. We can say that Hamiltonian for the system is

$$H(\vec{r}, \xi) = -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) + H_0(\xi) + V_{\text{coupl}}(\vec{r}, \xi),$$  \hspace{1cm} (4)$$

where $\mu$ is the reduced mass of the system, $V(r)$ is the bare potential in the absence of the coupling which consists of the nuclear and Coulomb parts ($V(r) = V_N(r) + V_C(r)$), $H_0(\xi)$ is the Hamiltonian for the intrinsic motion, and $V_{\text{coupl}}$ is the mentioned coupling. The Schrödinger equation for the total wave function then becomes

$$\left(-\frac{\hbar^2}{2\mu} \nabla^2 + V(r) + H_0(\xi) + V_{\text{coupl}}(\vec{r}, \xi)\right)\Psi(\vec{r}, \xi) = \epsilon \Psi(\vec{r}, \xi).$$  \hspace{1cm} (5)$$

The internal degree of freedom $\xi$ basically has a finite spin. We can write the coupling in multiples as

$$V_{\text{coupl}}(\vec{r}, \xi) = \sum_{\lambda, \alpha, \mu} f_\lambda(\vec{r})Y_{\lambda\mu}(\vec{r}).T_{3\lambda}(\xi),$$  \hspace{1cm} (6)$$

where $Y_{\lambda\mu}(\vec{r})$ is the spherical harmonics and $T_{3\lambda}(\xi)$ is the spherical tensors built from the internal coordinate. The dot means a scalar product. The sum is taken over all values of $\lambda$ except for $\lambda = 0$ since it is already considered in $V(r)$. The expansion basis for the wave function in Eq. (5) for a fixed total angular momentum $J$ and its $z$-component $M$ is defined as

$$(\hat{r}\xi|\langle nll|JM\rangle) = \sum_{m_l,m_I,m_m} (l\ell m_l)\langle nll|m_m\rangle Y_{m_l}(\hat{r})Y_{m_m}(\hat{\xi})\Phi_{nlm}(\xi),$$  \hspace{1cm} (7)$$

where $l$ and $\ell$ are the orbital and the internal angular momenta, respectively.

$\Phi_{nlm}(\xi)$ is the wave function for the internal motion which fulfills $H_0(\xi)\Phi_{nlm}(\xi) = \epsilon_n\Phi_{nlm}(\xi)$. The total wave function $\Psi(\vec{r}, \xi)$ is expanded with this basis as

$$\Psi(\vec{r}, \xi) = \sum_{n,l,I} \frac{u_{nl\ell}(r)}{r} (\hat{r}\xi|\langle nll|JM\rangle)$$  \hspace{1cm} (8)$$

The Schrödinger equation [Eq. (2)] can then be written as a set of coupled equations for $u_{nl\ell}(r)$

$$[\left(-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} - \frac{\epsilon_n}{r} + \frac{\epsilon_l}{r}\right) u_{nl\ell}(r) + \sum_{n',l',\ell'} V^I_{nl\ell,n'l'\ell'}(r) u_{nl'l'}(r) (r = 0),$$

where the coupling matrix elements $V^I_{nl\ell,n'l'\ell'}(r)$ are given by

$$V^I_{nl\ell,n'l'\ell'}(r) = \langle J M(nll) V_{\text{coupl}}(\vec{r}, \xi) (n'l', \ell')JM \rangle = \sum_{\lambda} (-1)^{l'-l+|\ell'|-\ell} f_{\lambda}(r)\langle \ell\ell'\ell'JM | V(r) | \ell\ell'\ell'JM \rangle$$

$$\times \sqrt{(2l+1)(2\ell+1)\frac{1}{2J+1}} \left\{ \begin{array}{ccc} I & \ell' & \ell \\ 0 & 0 & 0 \end{array} \right\}.$$  \hspace{1cm} (9)$$

The reduced matrix elements in Eq. (8) are defined by

$$\left\langle lm|Y_{\lambda\mu}(\ell')m'\right\rangle = \langle \ell'm'\lambda\mu|lm\rangle.$$  \hspace{1cm} (10)$$

Since $V^I_{nl\ell,n'l'\ell'}(r)$ are independent of the index $M$, the index has been suppressed as seen in (11). Equation (10) is called coupled-channels equations. For heavy-ion fusion reactions, these equations are usually solved using the incoming wave boundary conditions

$$u_{nl\ell}(r) \sim T_{nl\ell}(r)\exp\left[-i\frac{\epsilon_n}{r} k_{nl\ell}(r')\right] r \leq r_{abs},$$

$$r \to \infty (11)$$

where $k_{nl\ell} = \sqrt{2\mu} (E - \epsilon_n)/\hbar^2$, $k_{nl\ell}$ is $\sqrt{2\mu} E/\hbar^2$ and the local wave number $k_{nl\ell}$ is defined as

$$k_{nl\ell}(r) = \sqrt{\frac{2\mu}{\hbar^2}} (E - \epsilon_n - \frac{1}{2}(1+\lambda^2)\hbar^2/2\mu r^2 - V(r) - \frac{V^I_{nl\ell,n'l'\ell'}}{k_{nl\ell}}(r)), (12)$$

Once we get the transmission coefficients $T_{nl\ell}(r)$, the penetrability through the Coulomb barrier is given by

$$P_{nl\ell}(E) = \sum_{n'l'} \left| k_{nl\ell}/(2\mu)^{1/4} \right|^2 |T_{nl\ell}(r)|^2,$$  \hspace{1cm} (13)$$

$\lambda$ is the mentioned coupling. The Schrödinger equation for the entrance channel. The fusion cross section for unpolarized target is given by

$$\sigma_{fus}(r) = \frac{\pi}{k^2} \sum_{l\ell J I} \frac{2J+1}{2l+1} P_{l\ell J I}(E).$$  \hspace{1cm} (14)$$

The initial angular momentum $I_i$ is $J$ when the initial intrinsic spin $I_is$ zero. With the indexes $I_i$ and $I_f$ is suppressed in the penetrability, Eq. (17) then reads

$$\sigma_{fus}(E) = \pi/k^2 \sum_{J I} (2J+1) P_{J I}(E).$$  \hspace{1cm} (15)$$

$P_{J I}(E)$ is the penetrability is now affected by the channel couplings.

Contrary to the calculation of fusion cross sections, the calculation of quasi elastic cross sections often requires a large value of angular momentum in order to obtain converged results. The potential pocket at $r = r_{abs}$ becomes shallow or even disappears for such large angular momentum. Hence, the incoming flux in Eq. (13) cannot be properly identified. Therefore, the quasi-elastic problem usually employs the regular boundary conditions at the origin rather than using the incoming wave boundary conditions. When using the regular boundary conditions, a complex potential pocket at

$$\lambda = \frac{\epsilon_n}{r} \frac{\epsilon_l}{r} \frac{\epsilon}{r}$$

is obtained, the scattering amplitude can then be calculated as

$$f_{\lambda}(\theta, E) = -i \sum_{J I} \left| k_{nl\ell}/(2\mu)^{1/4} \right|^2 \left| \frac{\sigma_{I}^{I J} e^{i\phi_{I J}}}{2J+1} \right| \left| Y_{I J}(\theta) \right|^2,$$  \hspace{1cm} (16)$$

where $\sigma_{I}$ is the Coulomb phase shift and $f_{\lambda}$ is the Coulomb scattering amplitude. The phase shift is given by

$$\sigma_{I} = arg \Gamma(l + 1 + i\eta),$$  \hspace{1cm} (17)$$
where $\eta = Z_1 Z_2 e^2 / \hbar \nu$ is the Sommerfeld parameter, while the scattering amplitude is given by

$$f_c(\theta, E) = \frac{1}{n} \frac{m}{2} \sin^2(\theta/2) \exp \left[ -i \eta \ln(\sin^2(\theta/2)) + 2i \theta_0 (E) \right].$$

Using Eq. (16), the differential cross section is evaluated as

$$\frac{d\sigma_{qel}}{d\Omega} = \sum_{\mu \nu} \frac{k_{\mu \nu}}{k} |f_{\mu \nu}(\theta, E)|^2,$$

and from Eq. (18), the Rutherford cross section is given by

$$\frac{d\sigma_R}{d\Omega} = |f_c(\theta, E)|^2 = \frac{n^2}{4k^2} \sin^2(\theta/2).$$

3. Results

The coupled channel calculations were performed using a modified version of the computer code ccfull [9]. The chi square method $\chi^2$ is used to find the best fitted value of the diffuseness parameter in comparison with the experimental data. The experimental data are taken from ref. [15]. The data with $d\sigma_{qel}/d\Omega > 1$ are excluded from the fitting procedures, but included in the figures for completeness. This is because theoretically, it is clear that $d\sigma_{qel}/d\sigma_R$ cannot be larger than unity.

In our calculations, the nuclear potential has a real and an imaginary component. Both components are assumed to have Woods-Saxon forms. The purpose of the imaginary component is to simulate the compound nucleus formation. We use an imaginary potential with a potential depth of 30 MeV, a radius parameter of 1.0 fm, and a diffuseness parameter of 0.3 fm. The calculations are insensitive to the imaginary parameters provided that the imaginary potential is confined inside the Coulomb barrier. For the real part of the nuclear potential, the radius parameter $r_0$ is taken to be 1.22 fm. The value of potential depth $V_0$ depends on the diffuseness parameter such that the Coulomb barrier height $V_{th}$ for each system is reproduced. The calculations are carried out at scattering angle of $\theta_{m, m} = 170^\circ$. The radii of the target and projectile are taken as $r_T = r_T^{1/3} T$ and $r_P^{1/3} P$, respectively, where $r_T$ and $r_P$ are taken to be 1.2 fm in order to be consistent with the deformation parameters taken from ref. [11] and [12]. In order to ensure that the calculations are properly scaled according to the available experimental data, the calculated ratio of the quasi-elastic to the Rutherford cross sections for $a = 0.43$ fm (as the solid line) using a coupled-channel calculation at sub-barrier energies, where in this system, we are considered the projectile as inert with vibrational coupling for target nucleus. The best fitted diffuseness parameter is 0.43 fm, with $x^2 = 1.75$ and the potential depth $V_0 = 230.5$ MeV. The best fitted diffuseness parameter is considerably lower than the standard value (0.63 fm). However, from the resulting $\chi^2$ values, the best fitted diffuseness parameter obtained using a coupled-channels calculation (Projectile is vibrational channel and Target is an inert) fits the experimental data better than the one obtained through a coupled-channel calculation, where the target is vibrational channels and the Projectile is inert. Coupled–channels calculations using $a=0.45$ and 0.44 fm, respectively, are shown for comparison.

A. $^{48}$Ti + $^{208}$Pb system

In this system, we consider inert and vibrational coupling to the state 2$^+$ (0.983 MeV) for the projectile $^{48}$Ti nucleus with $\beta = 0.269$ [14]. The target $^{208}$Pb nucleus is considered to be inert and vibrational coupling with $\beta = 0.11$ to the state 3$^-$ (2.614 MeV). We use single-quadrupole and third-octupole phonon excitation in the projectile and target nucleus, respectively.

Channel couplings start to play an important role at energies above the sub-barrier region and therefore should be taken into account in our analyses here. Fig. 1 (a) shows the calculated ratio of the quasi-elastic to the Rutherford cross sections for $a = 0.43$ fm (as the solid line) using a coupled-channel calculation at sub-barrier energies, where in this system, we are considered the projectile as inert with vibrational coupling for target nucleus. The best fitted diffuseness parameter is 0.43 fm, with $x^2 = 1.75$ and the potential depth $V_0 = 230.5$ MeV. The best fitted diffuseness parameter is less than the standard value of around 0.63 fm.

The single phonon state of the quadrupole excitations into account, the best fitted diffuseness parameter obtained through a coupled-channels calculation is 0.44 fm. This is shown by the solid line in Fig. 2 (b). The $\chi^2$ value in comparison with the experimental data is 1.6, and the required potential depth $V_0$ is 233.5 MeV. The deduced diffuseness parameter is considerably lower than the standard value (0.63 fm). However, from the resulting $\chi^2$ values, the best fitted diffuseness parameter obtained using a coupled-channels calculation (Projectile is vibrational channel and Target is an inert) fits the experimental data better than the one obtained through a coupled-channel calculation, where the target is vibrational channels and the Projectile is inert. Coupled–channels calculations using $a=0.45$ and 0.44 fm, respectively, are shown for comparison.

![Figure 1](https://example.com/figure1.png)

**Figure 1:** The ratio of the quasi-elastic to the Rutherford cross sections for $^{48}$Ti + $^{208}$Pb system at sub-barrier energies. The analyses in the upper panel (a) are performed using coupled-channel with inert Projectile and vibrational channels, and the analyses in the lower panel (b) are performed using coupled-channels calculations with inert Target and vibrational Projectile channels. The experimental data are taken from Ref. [15]. The single-channel and coupled–channels calculations using $a=0.45$ and 0.46 fm, respectively, are shown for comparison.
B. $^{54}$Cr + $^{208}$Pb system

In this system, we consider inert and vibrational coupling to the state $^2\!^+\!^-$ (0.834 MeV) for the projectile $^{54}$Ti nucleus with $\beta=0.250$ [14]. The target $^{208}$Pb nucleus is considered to be inert and vibrational coupling with $\beta=0.11$ to the state $^3\!^-\!^-$ (2.614 MeV). We use double-quadrupole phonon excitation in the projectile nucleus.

For $^{54}$Cr + $^{208}$Pb system, the best fitted diffuseness parameter obtained using a coupled-channel calculation with inert Projectile (P) is 0.63 fm, with $\chi^2 = 4.46$ and $V_0 = 91.7$ MeV. The best fitted diffuseness parameter is in agreement with the standard value. The calculated ratio of the quasi-elastic to the Rutherford cross sections for using a coupled-channel calculation with inert P is shown by the solid line in Fig. 1 (a).

When a coupled-channel calculation with inert T and vibrational P is used, the best fitted diffuseness parameter is 0.67 fm, with $\chi^2 = 3.9$ and $V_0 = 91.7$ MeV. The best fitted diffuseness parameter, which is shown by the solid line in Fig. 2 (a), is higher than the standard value. Furthermore, the resulting $\chi^2$ values show that the best fitted diffuseness parameter obtained through a coupled-channels calculation with vibrational P and inert T fits the experimental data better than the one obtained through a coupled-channel calculation with inert T and vibrational T. The coupled-channels calculations using $a=0.65$ and 0.7 fm are shown in Fig. 2 (a and b) for comparison.

Figure 1: The ratio of the quasi-elastic to the Rutherford cross sections for $^{54}$Cr + $^{208}$Pb system at sub-barrier energies. The analyses in the upper panel (a) are performed using coupled-channel with inert P and vibrational T. And the analyses in the lower panel (b) are performed using coupled-channels calculations with inert T and vibrational P channels. The experimental data are taken from Ref. [10]. Coupled-channels calculations using $a=0.65$ and 0.70 fm, respectively, are shown for comparison.

C. $^{64}$Ni + $^{208}$Pb system

In this system, we consider inert and vibrational coupling to the state $^2\!^+\!^-\!^-$ (0.846 MeV) for the projectile $^{64}$Ni nucleus with $\beta=0.239$ [14]. The target $^{208}$Pb nucleus is considered to be inert and vibrational coupling with $\beta=0.11$ to the state $^3\!^-\!^-$ (2.614 MeV). We use triple-quadrupole phonon excitation in the projectile nucleus.

For $^{64}$Ni + $^{208}$Pb system, the best fitted diffuseness parameter obtained using a coupled-channel calculation with inert Projectile (P) is 0.63 fm, with $\chi^2 = 6.33$ and $V_0 = 74.9$ MeV. The best fitted diffuseness parameter is in agreement with the standard value. The calculated ratio of the quasi-elastic to the Rutherford cross sections for $a=0.63$ fm using a coupled-channel calculation with inert P and vibrational T is shown by the solid line in Fig. 3 (a).

When a coupled-channel calculation with vibrational P and T, the best fitted diffuseness parameter is 0.67 fm, with $\chi^2 = 3.9$ and $V_0 = 89.05$ MeV. The best fitted diffuseness parameter, which is shown by the solid line in Fig. 4.3(b), is higher than the standard value. Furthermore, the resulting $\chi^2$ values show that the best fitted diffuseness parameter obtained through a coupled-channels calculation fits the experimental data better than the one obtained through a coupled-channel calculation with inert P and vibrational T. The calculations using $a=0.63$ and 0.6 fm, are shown in Fig. 3 (a and b) for comparison. Also, a coupled-channel calculation inert T, the best fitted diffuseness parameter is 0.7 fm, with $\chi^2 = 6.33$ and $V_0 = 89.05$ MeV.

Figure 1: The ratio of the quasi-elastic to the Rutherford cross sections for $^{64}$Ni + $^{208}$Pb system at sub-barrier energies. The analyses in the upper panel (a) are performed using coupled-channel with inert P and vibrational T. And the analyses in the lower panel (b) are performed using coupled-channels calculations with inert T and vibrational P channels. The experimental data are taken from Ref. [10]. Coupled-channels calculations using $a=0.65$ and 0.70 fm, respectively, are shown for comparison.
performed using coupled-channels calculations with inert Target and vibrational Projectile channels. The experimental data are taken from Ref. [10] Coupled–channels calculations using $a=0.63$ and $0.60$ fm, respectively, are shown for comparison.

4. Conclusions

The nuclear potentials for some heavy-ion reactions have been studied through large-angle quasi-elastic scattering at sub-barrier energies of the $^{48}$Ti, $^{54}$Cr, and $^{56}$Fe + $^{208}$Pb systems. We have found that large-angle quasi-elastic scattering is a suitable method to study the diffuseness parameters of the nuclear potential.

The best fitted diffuseness parameters for $^{48}$Ti + $^{208}$Pb systems are significantly low compared to the standard value of 0.63 fm. But the best fitted diffuseness parameters for $^{54}$Cr + $^{208}$Pb systems are higher compared to the standard value of 0.63 fm. The calculated ratio of the quasi-elastic to the Rutherford cross sections for $^{48}$Ti and $^{54}$Cr + $^{208}$Pb systems give a good agreement using $a = 0.44$ fm and $a=0.67$ fm, respectively.

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6. References


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