On Weakly $f – \omega$ Continuous Functions

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Abstract: In this paper, we introduced a new class of functions called weakly $f - \omega$-continuous, $a\phi f - \omega$-continuous, $\beta f - \omega$-continuous, $p f - \omega$-continuous, $s f - \omega$-continuous, $a\phi f - \omega$-continuous, $\omega f - \omega$-continuous and $a\phi f - \omega$-continuous functions in fine-topological space and investigated some of their fundamental properties.

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Keywords: Topological spaces, $\omega$-open sets, $\omega$-closure, weakly $\omega$-continuous, fine-open, semi-continuous function, functions, semi-continuous function, fine-irresolute mappings which include pre-continuous sets etc. By using these fine-open sets they have defined $\beta - \omega$-sets namely fine-open sets which contains all space. In this space, they have defined a new class of open sets etc. introduced and studied by Sundaram and Sheik John (cf. [8], [9]) because generalized closed sets are only natural otherwise mentioned. For a subset $A$ of a space $(X, \tau)$ we say $A$ is $\omega$-closed if $\text{Cl}(A) = \text{Cl}(X, a) \cap \text{Int}(A))$. The complement of a semi-open set is called a semi-closed set [1].

1. Introduction

The notion of generalized closed sets has been studied extensively in recent years by many topologists [cf. [3], [7], [9]] because generalized closed sets are only natural generalization of closed sets. More importantly, they also suggest several new properties of topological spaces. A generalization of closed sets, $\omega$-closed sets were introduced and studied by Sundaram and Sheik John (cf. [8], [9]).

Powar P. L. and Rajak K. [4], have investigated a special case of generalized topological space called fine topological space. In this space, they have defined a new class of open sets namely fine-open sets which contains all $\alpha - \omega$-open sets, $\beta - \omega$-open sets, semi-open sets, pre-open sets, regular open sets etc. By using these fine-open sets they have defined fine-irresolute mappings which include pre-continuous functions, semi-continuous function, $\alpha - \omega$-continuous function, $\beta - \omega$-continuous functions, $\alpha - \text{irresolute functions}$, $\beta - \text{irresolute functions}$, etc.

In this paper, we introduced a new class of functions called weakly $f - \omega$-continuous, $a\phi f - \omega$-continuous, $\beta f - \omega$-continuous, $p f - \omega$-continuous, $s f - \omega$-continuous, $a\phi f - \omega$-continuous, $\omega f - \omega$-continuous and $a\phi f - \omega$-continuous functions in fine-topological space and investigated some of their fundamental properties.

2. Preliminaries

Throughout this paper, spaces always mean topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of space $(X, \tau)$, $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure of $A$ and the interior of $A$ in $X$, respectively. A subset $A$ of $X$ is said to be semi-open [2] if $A \subseteq \text{Cl}(\text{Int}(A))$. The complement of a semi-open set is called a semi-closed set [1].

Definition 2.1 Let $(X, \tau)$ be a topological space. A subset $A$ of $(X, \tau)$ is said to be $\omega$-closed in $(X, \tau)$ if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $X$. A subset $B$ of $(X, \tau)$ is said to be $\omega$-open if $X - B$ is $\omega$-closed (cf. [8]).

The family of all $\omega$-open (resp. $\omega$-closed) sets of $(X, \tau)$ is denoted by $\omega(X, \tau)$ (resp. $\omega(C(X, \tau))$ and sometimes they are denoted by $\omega(\tau)$ (resp. $\omega(C(\tau))$). Weset $\omega(X, x) = \{V \in \omega(\tau) : x \in V \}$ for $x \in X$. Note that the family of $\omega$-open subsets of $(X, \tau)$ forms a topology.

Definition 2.2 The union (resp. intersection) of all $\omega$-open (resp. $\omega$-closed) sets each contained in (resp. containing) a set $A$ in a space $X$ is called the $\omega$-interior (resp. $\omega$-closure) of $A$ and is denoted by $\omega(\text{Int}(A))$ (resp. $\omega(C(A))$) (cf. [9]).

Definition 2.3 A function $f : (X, \tau) \rightarrow (Y, \tau')$ is said to be $\omega$-continuous [8] if $f^{-1}(\omega(C(V))) = \omega(C(f^{-1}(V)))$ for every open set $V$ of $Y$ (resp. $V \in \omega(\tau')$).

Definition 2.4 A topological space $(X, \tau)$ is said to be $\omega$-regular [12] if for each closed set $F$ and each $x \in F$, there exist disjoint $\omega$-open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

Definition 2.5 A function $f : (X, \tau) \rightarrow (Y, \tau')$ is called weakly $\omega$-continuous if for each $x \in X$ and each open set $V$ containing $f(x)$ there exists $U \in \omega(O(X, x))$ such that $f(U) \subseteq \omega(C(V'))$ (cf. [5]).

Definition 2.6 A topological space $(X, \tau)$ is called $\omega$-connected [8] if $(X, \tau)$ cannot be written as the disjoint union of two nonempty $\omega$-open sets (cf. [9]).

Definition 2.7 A topological space $(X, \tau)$ is said to be $\omega$-T$_2$ [3] if for each pair of distinct points $x$ and $y$ in $X$, there exist $U \in \omega(O(X, x))$ and $V \in \omega(O(X, y))$ such that $U \cap V = \emptyset$ (cf. [5]).

Definition 2.8 Let $(X, \tau)$ be a topological space we define $\tau(\alpha) = \tau(\alpha)(say) = \{G_{\alpha}(\neq X) : G_{\alpha} \cap A_{\alpha} = \emptyset, \forall\alpha \in J\}$. The above collection $\tau_f$ of subsets of $X$ is called the fine collection of subsets of $X$ and $(X, \tau, \tau_f)$ is said to be the fine space $X$ generated by the topology $\tau$ on $X$ (cf. [4]).

Definition 2.9 A subset $U$ of a fine space $X$ is said to be a fine-open set of $X$ if $U$ belongs to the collection $\tau_f$ and the complement of every fine-open sets of $X$ is called the fine-closed sets of $X$ and we denote the collection by $F_f$ (cf. [4]).

Definition 2.10 Let $A$ be a subset of a fine space $X$, we say that a point $x \in X$ is a fine limit point of $A$ if every fine-open set of $X$ containing $x$ must contains at least one point of $A$ other than $x$ (cf. [4]).
Definition 2.11: Let $A$ be the subset of a fine space $X$, the fine interior of $A$ is defined as the union of all fine-open sets contained in the set $A$ i.e., the largest fine-open set contained in the set $A$ and is denoted by $\text{fin int}(A)$ (cf. [4]).

Definition 2.12: Let $A$ be the subset of a fine space $X$, the fine closure of $A$ is defined as the intersection of all fine-closed sets containing the set $A$ i.e., the smallest fine-closed set containing the set $A$ and is denoted by $\text{fin cl}(A)$ (cf. [4]).

Definition 2.13: A function $f$: $(X, \tau, \tau_f)$ $\rightarrow$ $(Y, \tau', \tau'_f)$ is called fine-irresolute (or $f$-irresolute) if $f^{-1}(V)$ is fine-open in $X$ for every fine-open set $V$ of $Y$ (cf. [4]).

Definition 2.14: A fine-open set $S$ of a fine space $(X, \tau, \tau_f)$ (cf. [4]) is called

1. $\alpha f$ -open if $S$ is an open subset of a topological space $(X, \tau)$.
2. $\beta f$ -open if $S$ is a $\beta$-open subset of a topological space $(X, \tau)$.
3. $\gamma f$ -open if $S$ is a $\gamma$-open subset of a topological space $(X, \tau)$.
4. $\delta f$ -open if $S$ is a $\delta$-open subset of a topological space $(X, \tau)$.

Definition 2.15: A function $f$: $(X, \tau, \tau_f)$ $\rightarrow$ $(Y, \tau', \tau'_f)$ is said to be $f$-irresolute if $f^{-1}(V)$ is $\alpha f$ -open in $X$ for every $\alpha f$ -open set $V$ of $Y$ (cf. [4]).

Definition 2.16: A function $f$: $(X, \tau, \tau_f)$ $\rightarrow$ $(Y, \tau', \tau'_f)$ is said to be $\beta f$-irresolute if $f^{-1}(V)$ is $\beta f$ -open in $X$ for every $\beta f$ -open set $V$ of $Y$ (cf. [4]).

Definition 2.17: A function $f$: $(X, \tau, \tau_f)$ $\rightarrow$ $(Y, \tau', \tau'_f)$ is said to be $\gamma f$-irresolute if $f^{-1}(V)$ is $\gamma f$ -open in $X$ for every $\gamma f$ -open set $V$ of $Y$ (cf. [4]).

Definition 2.18: A function $f$: $(X, \tau, \tau_f)$ $\rightarrow$ $(Y, \tau', \tau'_f)$ is said to be $\delta f$-irresolute if $f^{-1}(V)$ is $\delta f$ -open in $X$ for every $\delta f$ -open set $V$ of $Y$ (cf. [4]).

3. Weakly $f - \omega$- Continuous Functions

In this section, we define weakly $f - \omega$-continuous functions in fine-topological space.

Definition 3.1: Let $(X, \tau, \tau_f)$ be a fine-topological space. A subset $A$ of $(X, \tau, \tau_f)$ is said to be $f - \omega$-closed in $(X, \tau, \tau_f)$ if $f \omega(A) \subset U$ whenever $A \subset U$ and $U$ is $f$-semi-open in $X$. A subset $B$ of $(X, \tau, \tau_f)$ is said to be $f - \omega$-open if $X - B$ is $f - \omega$-closed.

The family of all $f - \omega$-open (resp. $f - \omega$-closed) sets of $(X, \tau, \tau_f)$ is denoted by $\omega(X, \tau, \tau_f)$ (resp. $\omega f(X, \tau, \tau_f)$) and sometimes they are denoted by $\omega f(\tau_f)$ (resp. $\omega f(\tau_f)$). Weset $\omega(X, x) = \{V \in \omega(\tau_f)|x \in V\}$ for $x \in X$. Note that the family of $f - \omega$-open subsets of $(X, \tau, \tau_f)$ forms a topology.

Example 3.1: Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, c\}, \{a, b\}\}$.

It may be easily checked that the set $(b, c)$ is $f - \omega$-open.

Definition 3.2: The union (resp. intersection) of all $f - \omega$-open (resp. $f - \omega$-closed) sets each contained in (resp. containing) a set $A$ in a space $X$ is called the $f - \omega$-interior (resp. $f - \omega$-closure) of $A$ and is denoted by $f \omega int(A)$ (resp. $f \omega cl(A)$).

Definition 3.3: A function $f$: $(X, \tau, \tau_f)$ $\rightarrow$ $(Y, \tau', \tau'_f)$ is said to be $f - \omega$-continuous if $f^{-1}(V) \in \omega(\tau_f)$ for every open set $V$ of $Y$.

Example 3.2: Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$.

It may be easily checked that, the only $f - \omega$-open sets of $Y$ are $\phi$, $Y$, $\{1, 1, 2\}$, and their respective pre-images are $\phi$, $X$, $\{a, b\}$, which are $f - \omega$-open in $X$. Thus, $f$ is $f - \omega$-continuously.

Definition 3.4: A function $f$: $(X, \tau, \tau_f)$ $\rightarrow$ $(Y, \tau', \tau'_f)$ is said to be $f - \omega$-irresolute if $f^{-1}(V) \in \omega(\tau_f)$ for every $V \in \omega(\tau_f)$.

Example 3.3: Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$.

We define a map $f$: $(X, \tau_f)$ $\rightarrow$ $(Y, \tau'_f)$ by $f(a) = 1, f(b) = 2, f(c) = 3$. It may be easily checked that, the only $f - \omega$-open sets of $Y$ are $\phi$, $Y$, $\{1, 1, 2\}$, and their respective pre-images are $\phi$, $X$, $\{a, b\}$, $\{a, c\}$, which are $f - \omega$-open in $X$. Thus, $f$ is $f - \omega$-irresolute.

Definition 3.5: A function $f$: $(X, \tau) \rightarrow (Y, \tau')$ is called weakly $f - \omega$-continuous for each $x \in X$ and each open set $V$ containing $f(x)$ there exists $U \in \omega f(X, \tau')$ such that $f(U) \in \omega f(V')$.

Example 3.4: Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$.

We define a map $f$: $(X, \tau_f)$ $\rightarrow$ $(Y, \tau'_f)$ by $f(a) = 1, f(b) = 2, f(c) = 3$. It may be easily checked that, the map $f$ is weakly $f - \omega$-continuous.

Example 3.5: Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $\mu' = \{\phi, \{a\}, X\}$. Then the identity function $f$: $(X, \tau_f, \tau'_f)$ $\rightarrow$ $(X, \mu, \mu')$ is weakly $\omega$-continuous but not $\omega$-continuous.

Definition 3.6: A function $f$: $(X, \tau, \tau_f)$ $\rightarrow$ $(Y, \tau', \tau'_f)$ is said to be fine-semi-$\omega$-continuous if $f^{-1}(V)$ is $\omega$-open in $X$ for every semi-open set $V$ of $Y$.

Definition 3.7: A function $f$: $(X, \tau, \tau_f)$ $\rightarrow$ $(Y, \tau', \tau'_f)$ is said to be quasi-$\omega$-continuous if $f^{-1}(V)$ is $\alpha f$ -open in $X$ for every $\omega$-open set $V$ of $Y$.
Definition 3.8 A function \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) is said to be \( \beta \beta \omega \)-continuous if \( f^{-1}(V) \) is \( \beta f \)-open in \( X \) for every \( \omega \)-open set \( V \) of \( Y \).

Definition 3.9 A function \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) is said to be \( \beta \beta \omega \)-continuous if \( f^{-1}(V) \) is \( \beta p f \)-open in \( X \) for every \( \omega \)-open set \( V \) of \( Y \).

Definition 3.10 A function \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) is said to be \( \beta \omega \)-continuous if \( f^{-1}(V) \) is \( \omega \)-open in \( X \) for every \( \omega \)-open set \( V \) of \( Y \).

Definition 3.11 A function \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) is said to be \( \beta \omega \)-continuous if \( f^{-1}(V) \) is \( \omega \)-open in \( X \) for every \( \alpha f \)-open set \( V \) of \( Y \).

Definition 3.12 A function \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) is said to be \( \omega \phi \)-continuous if \( f^{-1}(V) \) is \( \beta \phi \omega \)-open in \( X \) for every \( \beta \phi \omega \)-open set \( V \) of \( Y \).

Definition 3.13 A function \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) is said to be \( \beta \omega \)-continuous if \( f^{-1}(V) \) is \( \omega \)-open in \( X \) for every \( \omega \)-open set \( V \) of \( Y \).

Definition 3.14 A function \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) is said to be \( \omega \phi \)-continuous if \( f^{-1}(V) \) is \( \beta \phi \omega \)-open in \( X \) for every \( \beta \phi \omega \)-open set \( V \) of \( Y \).

Theorem 3.1 Let \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) and \( g: (Y, \tau', \tau_f') \to (Z, \tau', \tau_f') \) be functions. Then the composition \( g \circ f \) is \( \alpha \omega \)-continuous if \( f \) is \( \alpha \omega \)-continuous and \( g \) is \( \omega \phi \)-continuous.

Proof. Let \( V \) be any \( \omega \)-open subset of \( Z \). Since \( g \) is \( \omega \phi \)-continuous, \( g^{-1}(V) \) is \( \omega \)-open in \( Y \). Since \( \beta \phi \omega \)-continuous is \( \beta \phi \omega \)-continuous, \( f^{-1}(g^{-1}(V)) \) is \( \beta \phi \omega \)-open in \( X \). Thus, \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is \( \beta \phi \omega \)-continuous.

Theorem 3.2 Let \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) and \( g: (Y, \tau', \tau_f') \to (Z, \tau', \tau_f') \) be functions. Then the composition \( g \circ f \) is \( \beta \phi \)-continuous if \( f \) is \( \beta \phi \)-continuous and \( g \) is \( \omega \phi \)-continuous.

Proof. Similar to Theorem 3.1.

Theorem 3.3 Let \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) and \( g: (Y, \tau', \tau_f') \to (Z, \tau', \tau_f') \) be functions. Then the composition \( g \circ f \) is \( \beta \phi \)-continuous if \( f \) is \( \beta \phi \)-continuous and \( g \) is \( \omega \phi \)-continuous.

Proof. Similar to Theorem 3.1.

Theorem 3.4 Let \( f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f') \) and \( g: (Y, \tau', \tau_f') \to (Z, \tau', \tau_f') \) be functions. Then the composition \( g \circ f \) is \( \omega \phi \)-continuous if \( f \) is \( \omega \phi \)-continuous and \( g \) is \( \omega \phi \)-continuous.

Proof. Similar to Theorem 3.1.
Proof. Suppose that \((Y, r')\) is not fine-connected. Then there exist nonempty disjoint fine-open sets \(V_1\) and \(V_2\) in \(Y\) such that \(V_1 \cup V_2 = Y\). Since \(f\) is surjective, \(f^{-1}(V_1)\) and \(f^{-1}(V_2)\) are nonempty disjoint subsets of \(X\) such that \(f^{-1}(V_1) \cup f^{-1}(V_2) = X\). By Theorem 3.1, we have \(f^{-1}(V_i) \subseteq \omega_{int}(f^{-1}(I_{cl}(V_i))), i = 1, 2\). Since \(V_i\) is fine-open and closed and every fine-closed set is \(f\omega\)-closed, we obtain \(f^{-1}(V_i) \subseteq \omega_{int}(f^{-1}(V_i))\) and hence \(f^{-1}(V_i)\) is \(f\omega\)-open for \(i = 1, 2\). This implies that \((X, r)\) is not \(f\omega\)-connected. This shows that if \(X\) is \(f\omega\)-connected, then \(Y\) is fine-connected.

5. Conclusion

By using the concepts of \(f\omega\)-closed sets on fine-topological space, we may define a generalized form of continuity i.e. called weakly \(f\omega\)-continuous, \(\alpha f\omega\) -continuous, \(\beta f\omega\) -continuous, \(p f\omega\) -continuous, \(sf\omega\) -continuous, \(\omega sf\) -continuous, \(\alpha pf\) -continuous and \(\alpha opf\) -continuous in fine-topological space and investigated some of their fundamental properties. Also, by defining some irresolute maps, the more general form of homeomorphism can be studied which is widely used in quantum physics.

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