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Analysis to Thermoelastic Interactions under a Heat Conduction Model with a Delay Term

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Abstract: ***The present work is concerned with a heat conduction model: an exact heat conduction model with a delay term. A generalized thermoelasticity theory was proposed by Roychoudhuri (2007) based on the heat conduction law with three -phase-lag effects for the purpose of considering the delayed response in times due to the microstructural interactions in the heat transport mechanism. However, the model define ill-posed problem. Hence, Recently, Quintanilla (2011) has proposed to reformulate this constitutive equation as a heat conduction theory with a single deley term and has investigated the spatial behavior of the solutions for this theory. A Phragmen- Lindelof type alternative is obtained and it has been shown that the solutions either decay in an exponential way or blow-up at infinity in an exponential way. The obtained results are extended to a thermoelasticity theory by considering the Taylor series approximation of the equation of heat conduction to the delay term and Phragmen-Lindelof type alternative is obtained for the forward and backward in time equations. The formulation is then is applied to solve a boundary value problem of an isotropic elastic half space with its plane boundary subjected to exponential decrease in temperature and zero stress. The homotopy analysis method is applied to obtain the solution of the problem.. The problem is discussed with different graphs of numerical values of the field variables.***

Keywords: Thermoelasticity, Relaxation Parameter, Homotopy, Half Space

1. Introduction

The linear theory of heat conduction suggests that any thermal disturbance at some point in a material body will be felt instantaneously at all points of the body. It shows a physically unrealistic situation as it implies that thermal wave propagates with infinite speed. Classical coupled dynamical theory of thermoelasticity introduced by Biot (1956) is based on this classical heat conduction model and suffers from the same drawback that the thermal wave propagates with infinite speed although the elastic wave propagates with a finite speed under this theory. Because of it's causality important role in modern physics, different non-classical heat conduction theories and also various thermoelasticity theories are proposed over the 20th century to eliminate this physical drawback. A systematic development in the heat conduction theory and thermoelasticity theory can be found out in the review article [08] which have reported several studies concerning applicability of these non-classical thermoelasticity theories. We recall the pioneering contributions by Lord and Shulman (1967) and Green and Lindsay (1972). In the Lord-Shulman model (1967), the heat conduction law is replaced with the Cattaneo-Vernotte heat conduction model that includes one thermal relaxation time parameter, where as in the model proposed by Green and Lindsay (1972), an explicit version of the constitutive equations involved in the coupled thermoelasticity theory is introduced. All the equations of this coupled theory are modified by including two parameters into the constitutive equations that act as thermal relaxation time parameters. Later on, another thermoelasticity theory is proposed by Green and Naghdi [4-6] by developing an alternative formulation of heat propagation. They incorporated the thermal pulse transmission in a very logical manner. This theory is divided into three parts, namely, GN-I, GN-II and GN-III. In this theory, the thermal displacement (υ) and gradient of thermal displacement $(\nabla\upsilon)$ are introduced as new constitutive variables, where: $\dot{\upsilon}=T, T$ being the temperature. In the formulation of this theory, the usual entropy production inequality is replaced with an entropy balance law (see also Chandrasekharaiah (1998)) and the heat conduction law for GN-III model is proposed in the form

$$q = -[\kappa\nabla T + \kappa^{*}\nabla\upsilon]$$

where, q is the heat flux vector and κ is thermal conductivity. The positive constant κ^{*} is a material parameter (conductivity rate of the material) newly introduced and is considered as the characteristic of the theory. In 1995, Tzou has incorporated the effects of microstructural interactions in the fast transient process of heat transport phenomenon and proposed a more generalized relation for heat conduction, known as dual phase-lag model, in the form

$$q(P, t+\tau_q) = -\kappa\left[\nabla T(P, t+\tau_T)\right]$$

This constitutive law incorporates two delay times, called as phase-lags. The first delay time τ_q is denoted as the phase lag of the heat flux vector and the second delay time (τ_T) is termed as the phase-lag of the temperature gradient that captures the effect of phonon-electron interactions, a microscale response in space during heat transport mechanism. Both the phase-lags are assumed to be positive and they are the intrinsic properties of the medium (Tzou, (1995)). Subsequently, dual phase-lag thermoelastic theory is developed by Tzou and Chandrasekharaiah. Later on, a three phase-lag thermoelasticity theory is introduced by Roychoudhuri (2007) in the frame of thermoelasticity theory by employing the three phase-lag heat conduction constitutive relation in place of the above relation given by

Green and Naghdi. The constitutive relation for heat-flux vector in the three phase-lag theory is given in the form

$$q(P,t+\tau_q) = -\kappa\left[\nabla T(P,t+\tau_T)+\nabla \upsilon(P,\upsilon+\tau_\upsilon)\right]$$

Here τ_υ is the additional delay time known as phase-lag of thermal displacement gradient. The above mentioned thermoelasticity theories have drawn the serious attention of researchers in recent years and investigated several features of various models. Some critical analysis on these models are also reported. For example, Dreher (2009) have reported a critical analysis on dual-phase-lag and three phase-lag heat conduction model and shown that when we adjoin this constitutive equation with energy equation

$$-\nabla q(x,t)= c\dot{T}(x,t)$$

The present work is concerned with a very recently proposed heat conduction model: an exact heat conduction model with a delay term. A generalized thermoelasticity theory was proposed by Roychoudhuri (2007) based on the heat conduction law with three -phase-lag effects for the purpose of considering the delayed response in times due to the microstructural interactions in the heat transport mechanism. However, the model define ill-posed problem. Hence, Recently, Quintanilla (2011) has proposed to reformulate this constitutive equation as a heat conduction theory with a single deley term and has investigated the spatial behavior of the solutions for this theory. A Phragmen- Lindelof type alternative is obtained and it has been shown that the solutions either decay in an exponential way or blow-up at infinity in an exponential way. The obtained results are extended to a thermoelasticity theory by considering the Taylor series approximation of the equation of heat conduction to the delay term and Phragmen-Lindelof type alternative is obtained for the forward and backward in time equations. The formulation is then is applied to solve a boundary value problem of an isotropic elastic half space with its plane boundary subjected to exponential decrease in temperature and zero stress. The homotopy analysis method is applied to obtain the solution of the problem.. The problem is discussed with different graphs of numerical values of the field variables.

2. Governing Equations

We employ the thermoelasticity theory based on the exact heat conduction model with a delay as proposed by Laseduarte and Qunitanilla [10] to consider the thermoelastic interactions in a homogeneous isotropic solid in the absence of body forces and heat sources. The basic governing equations in the vector form, therefore can be written in a unified way as follows:

(I) Equation of motion

$$\mu\nabla^2 u+(\lambda+\mu)\vec{\nabla}\, div.u-\gamma\nabla T=\rho\ddot{u} \tag{1}$$

(II) Heat conduction equation

$$\kappa\nabla^2\dot{T}+\kappa^*\left[\nabla^2 T+\tau\nabla^2\dot{T}+\frac{\tau^2}{2}\nabla^2\ddot{T}\right] = \rho C_E\ddot{T}+\gamma T_0 div.\ddot{u} \tag{2}$$

(III) Stress–displacement–temperature relations

$$\sigma=\lambda(div.u)\vec{I}+\mu\left(\vec{\nabla}u+\vec{\nabla}u^T\right)-\gamma T\vec{I} \tag{3}$$

Where, where, u is the displacement vector, T is the temperature, $\vec{\sigma}$ is the stress tensor, $\vec{I}$ is the identity tensor, λ,μ are Lame's constants, ρ is the density and γ is the coefficient of linear thermal expansion. κ is the thermal conductivity, κ^* is the thermoelatic constant, τ is thermal relaxation parameter and T_0 is the initial temperature.

3. Formulation of the Problem

We employ the thermoelasticity theory based on the exact heat conduction model with a delay term as proposed by Leseduarte and Qunitanilla [10] to consider the thermoelastic interactions in a homogeneous and isotropic solid in the absence of body forces and heat sources. The equation of motion, stress-strain-temperature relations and the unified heat conduction equation in the contexts of the theory of extended thermoelasticity due to Lord and Shulman (LS model), thermoelasticity without energy dissipation due to GN-II model, thermoelasticity of type III (GN-III model) and the new model with a delay parameter as proposed by Leseduarte and Qunitanilla [10] can be considered in a unified way as follows:

We will use the Cartesian co-ordinates (x,y,z) and the components of displacements $u_i=(u,0,0)$ and write them as follows

$$\frac{\partial^2 u}{\partial t^2}=(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}-\gamma\frac{\partial T}{\partial x} \tag{4}$$

$$\kappa\frac{\partial^2 T}{\partial x^2}=\left(\frac{\partial}{\partial t}+\tau\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T+\gamma T_0 e\right) \tag{5}$$

$$\sigma=(\lambda+2\mu)\frac{\partial u}{\partial x}-\gamma(T-T_0) \tag{6}$$

where $e=\dfrac{\partial u}{\partial x}$

To transform the above equations in non-dimensional forms, we define the following non-dimensional variables

$$x'=c_1\eta x,\quad u'=c_1\eta u,\quad t'=c_1^2\eta t,$$

$$\tau'=c_1^2\eta\tau,\quad \theta'=\frac{\gamma(T-T_0)}{\lambda+2\mu}$$

$$\sigma'_{xx}=\frac{\sigma_{xx}}{\lambda+2\mu},\quad c_1^2=\frac{\lambda+2\mu}{\rho},\quad \eta=\frac{\rho C_E}{\kappa}$$

Equations (4)-(6) then reduce to the following non-dimensional forms (the primes are dropped for convenience)

$$\frac{\partial^2 u}{\partial x^2} - a_1 \frac{\partial \theta}{\partial x} = \frac{\partial^2 u}{\partial t^2} \quad (7)$$

$$\frac{\partial^3 \theta}{\partial t \partial x^2} + \frac{1}{4}\left[\frac{\partial^2 \theta}{\partial x^2} + \tau \frac{\partial^3 \theta}{\partial t \partial x^2} + \frac{\tau^2}{2}\frac{\partial^4 \theta}{\partial t^2 \partial x^2}\right] = \frac{\partial^2 \theta}{\partial t^2} + a_2 \frac{\partial^3 u}{\partial t^2 \partial x} \quad (8)$$

$$\sigma = \frac{\partial u}{\partial x} - a_1 \theta \quad (9)$$

Where, $a_1 = \frac{\gamma T_0}{\lambda + 2\mu}, a_2 = \frac{\gamma}{\rho c_v}$.

4. Solutions Using The Homotopy Perturbation Method

From (7) and (9), we get,

$$\frac{\partial^2 \sigma}{\partial x^2} = \frac{\partial^2 \sigma}{\partial t^2} + a_1 \frac{\partial^2 \theta}{\partial t^2} \quad (10)$$

Using equation (7), (8) and (9), we get,

$$\frac{\partial^2 \theta}{\partial x^2} = 4a_2 \frac{\partial^2 \sigma}{\partial t^2} + \left[4(1-\varepsilon)\frac{\partial^2 \theta}{\partial t^2} - (4+\tau)\frac{\partial^3 \theta}{\partial t \partial x^2} - \frac{\tau^2}{2}\frac{\partial^4 \theta}{\partial t^2 \partial x^2}\right] \quad (11)$$

Where, $\varepsilon = a_1 a_2$.

Equations (10) and (11) represents the second order coupled thermoelastic equations in temperature θ and stress σ .

Let us assume boundary conditions as follows,

$$\left.\begin{aligned} \theta(t,0) &= 2 - e^{-t} \\ \sigma(t,0) &= 0 \end{aligned}\right\} \quad (12)$$

According to HPM (S.J. Liao [12]), we construct the following simple homotopies:

$$\frac{\partial^2 \sigma}{\partial x^2} + p\left[-\frac{\partial^2 \sigma}{\partial t^2} - a_1 \frac{\partial^2 \theta}{\partial t^2}\right] = 0 \quad (13)$$

$$\frac{\partial^2 \theta}{\partial x^2} + p[-4a_2 \frac{\partial^2 \sigma}{\partial t^2} - \{4(1-\varepsilon)\frac{\partial^2 \theta}{\partial t^2} - (4+\tau)\frac{\partial^3 \theta}{\partial t \partial x^2} - \frac{\tau^2}{2}\frac{\partial^4 \theta}{\partial t^2 \partial x^2}\}] = 0 \quad (14)$$

Where $p \in [0,1]$ is an embedding parameter, we use it to expand the solution in the following form:

$$\sigma(x,t) = \sigma_0 + p\sigma_1 + p^2\sigma_2 + p^3\sigma_3 + \ldots \quad (15)$$

$$\theta(x,t) = \theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \ldots \quad (16)$$

The approximate solution can be obtained by setting $p = 1$ in equation (15) and (16).

Now, substituting $\sigma(x,t)$ and $\theta(x,t)$ from (15), (16) to (13), (14) respectively, and equating the terms with identical powers of p, we can obtained series of linear equations. These linear equations are easy to solve by using Mathematica software to get as many equations as we need in the calculation of the numerical as well as explicit solutions. Here we write only the first few linear equations:

$$p^0: \quad \frac{\partial^2 \sigma_0}{\partial x^2} = 0 \quad (17a)$$

$$\frac{\partial^2 \theta_0}{\partial x^2} = 0 \quad (17b)$$

$$p^1: \quad \frac{\partial^2 \sigma_1}{\partial x^2} = \frac{\partial^2 \sigma_0}{\partial t^2} + a_1 \frac{\partial^2 \theta_0}{\partial t^2} \quad (18a)$$

$$\frac{\partial^2 \theta_1}{\partial x^2} = 4a_2 \frac{\partial^2 \sigma_0}{\partial t^2} + \{4(1-\varepsilon)\frac{\partial^2 \theta_0}{\partial t^2} - (4+\tau)\frac{\partial^3 \theta_0}{\partial t \partial x^2} - \frac{\tau^2}{2}\frac{\partial^4 \theta_0}{\partial t^2 \partial x^2}\} \quad (18b)$$

$$p^2: \quad \frac{\partial^2 \sigma^2_{xx}}{\partial x^2} = \frac{\partial^2 \sigma^1_{xx}}{\partial t^2} + a_1 \frac{\partial^2 \theta^1}{\partial t^2} \quad (19a)$$

$$\frac{\partial^2 \theta_2}{\partial x^2} = 4a_2 \frac{\partial^2 \sigma_1}{\partial t^2} + [4(1-\varepsilon)\frac{\partial^2 \theta_1}{\partial t^2} - (4+\tau)\frac{\partial^3 \theta_1}{\partial t \partial x^2} - \frac{\tau^2}{2}\frac{\partial^4 \theta_1}{\partial t^2 \partial x^2}] \quad (19b)$$

The solution of equation (17) can be calculated by using the boundary conditions (12):

$$\begin{aligned} \sigma_0(x,t) &= \sigma(t,0) = 0, \\ \theta_0(x,t) &= \theta(t,0) = 2\text{-}e^{-t} \end{aligned} \quad (20)$$

Then, we can derive the solution of (18) in the following form,

$$\sigma_1(x,t) = \int_0^x \int_0^x \frac{\partial^2 \sigma_0(v,t)}{\partial t^2} dv\, dv + a_1 \int_0^x \int_0^x \frac{\partial^2 \theta_0(v,t)}{\partial t^2} dv\, dv$$

$$\therefore \quad \sigma_1(x,t) = -a_1\left(\frac{x^2}{2!}\right)e^{-t} \quad (21)$$

$$\theta_1(x,t) = 4a_2 \int_0^x \int_0^x \frac{\partial^2 \sigma_0(v,t)}{\partial t^2} dvdv + \int_0^x \int_0^x [4(1-\varepsilon)\frac{\partial^2 \theta_1(v,t)}{\partial t^2} - (4+\tau)\frac{\partial^3 \theta_1(v,t)}{\partial t \partial x^2} - \frac{\tau^2}{2}\frac{\partial^4 \theta_1(v,t)}{\partial t^2 \partial x^2}]dvdv$$

$$\therefore \quad \theta_1(x,t) = -4(1-\varepsilon)\left(\frac{x^2}{2!}\right)e^{-t} \quad (22)$$

Similarly,

$$\sigma_2(x,t) = -a_1(3-\varepsilon)\frac{x^4}{4!}e^{-t} \quad (23)$$

$$\theta_2(x,t) = -4\left[\varepsilon + 4(1-\varepsilon)^2\right]\frac{x^4}{4!}e^{-t} -4(1-\varepsilon)\left[4+\tau-\frac{\tau^2}{2}\right]\frac{x^2}{2!}e^{-t} \quad (24)$$

$$\sigma_3(x,t) = -a_1\left[3(1+\varepsilon)+4^2(1-\varepsilon)^2\right]\frac{x^6}{6!}e^{-t} - 4a_1(1-\varepsilon)\left(4+\tau-\frac{\tau^2}{2}\right)\frac{x^4}{4!}e^{-t} \quad (25)$$

$$\theta_3(x,t) = -4\left[\varepsilon(3-\varepsilon)-4(1-\varepsilon)\left\{\varepsilon+4(1-\varepsilon)^2\right\}\right]\frac{x^6}{6!}e^{-t} -4[4(1-\varepsilon)^2\left(4+\tau-\frac{\tau^2}{2}\right)+(4+\tau)\left\{\varepsilon+4(1-\varepsilon)^2\right\} -\frac{\tau^2}{2}\left\{\varepsilon+4(1-\varepsilon)^2\right\}]\frac{x^4}{4!}e^{-t}+4(1-\varepsilon)\left(4+\tau-\frac{\tau^2}{2}\right)^2\frac{x^2}{2!}e^{-t} \quad (26)$$

Therefore the complete approximate solution can be readily obtained by the same iterated process using the Mathematica package.

5. Truncate The Series Solution Obtained By HPM:

$$\sigma(x,t) \cong \tilde{\sigma}(x,t) = \sum_{i=0}^{3}\sigma_i(x,t) = \left[m_1\frac{x^2}{2!}+m_2\frac{x^4}{4!}+m_3\frac{x^6}{6!}\right]e^{-t} \quad (27)$$

$$\theta(x,t) \cong \tilde{\theta}(x,t) = \sum_{i=0}^{3}\theta_i(x,t) = 2+[-1+n_1\frac{x^2}{2!}+n_2\frac{x^4}{4!}+n_3\frac{x^6}{6!}]e^{-t} \quad (28)$$

From equation (9), we find that

$$\frac{\partial u}{\partial x} = [a_1+(m_1+a_1n_1)\frac{x^2}{2!} +(m_2+a_1n_2)\frac{x^4}{4!}+(m_3+a_1n_3)\frac{x^6}{6!}]e^{-t}$$

Now, integrating above equation with respect to x, we get

$$u(x,t) = [\frac{a_1x}{1!}+(m_1+a_1n_1)\frac{x^3}{3!} +(m_2+a_1n_2)\frac{x^5}{5!}+(m_3+a_1n_3)\frac{x^7}{7!}]e^{-t} \quad (29)$$

Where,

$$m_1 = -a_1,\quad m_2 = -a_1\left[(3-\varepsilon)+4(1-\varepsilon)\left(4+\tau-\frac{\tau^2}{2}\right)\right]$$

$$m_3 = -a_1\left[3(1+\varepsilon)+4^2(1-\varepsilon)^2\right]$$

$$n_1 = 4(1-\varepsilon)\left[\left(4+\tau-\frac{\tau^2}{2}\right)^2-\left(4+\tau-\frac{\tau^2}{2}\right)-1\right]$$

$$n_2 = -4\left\{\varepsilon+4(1-\varepsilon)^2\right\}-4[4(1-\varepsilon)^2\left(4+\tau-\frac{\tau^2}{2}\right) +(4+\tau)\left\{\varepsilon+4(1-\varepsilon)^2\right\}-\frac{\tau^2}{2}\left\{\varepsilon+4(1-\varepsilon)^2\right\}$$

$$n_3 = -4\left[\varepsilon(3-\varepsilon)-4(1-\varepsilon)\left\{\varepsilon+4(1-\varepsilon)^2\right\}\right]$$

6. Numerical Results

In this section, we illustrate the problem with numerical values of the field variables like displacement, temperature and stress for a material in space-time domain. Therefore, with the help of Mathematica software, we compute values of the variables by using solutions obtained in (24)-(26) and employing the numerical method. We choose the copper material for this purpose and physical data for it are given as follows:

$$\lambda = 7.76\times10^{10}\,Nm^{-2}, \mu = 3.86\times10^{10}\,Nm^{-2},$$
$$\alpha_t = 1.78\times10^{-5}K^{-1}, \kappa = 8886.73\,sm^{-2},$$
$$c_v = 383.1\,JKg^{-1}K^{-1}, \rho = 8954\,Kgm^{-3}.$$

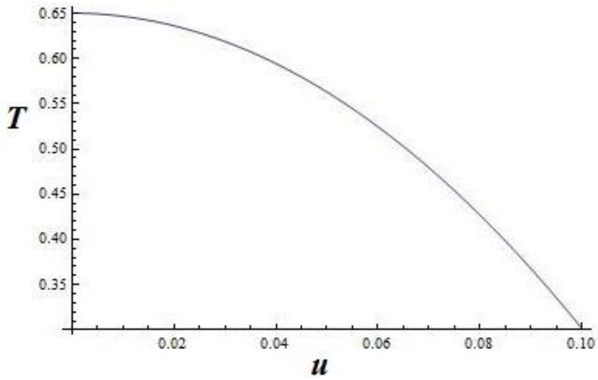

Figure 1: Variation of temperature, T vs. x at $t = 0.3$.

x

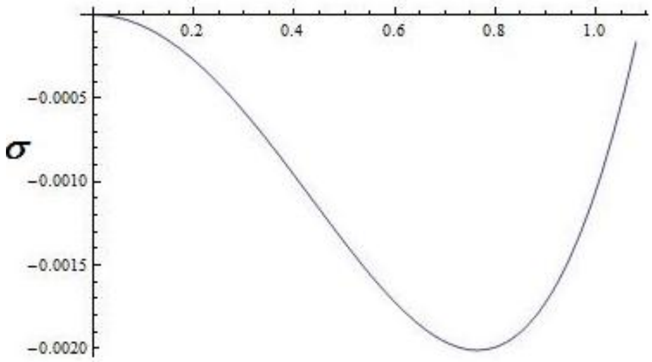

Figure 2: Variation of Stress, σ vs. x at $t = 0.3$.

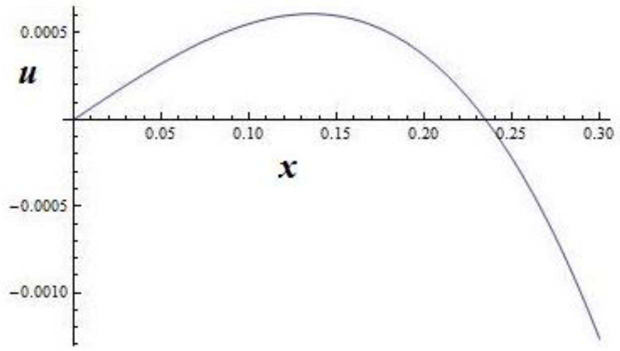

Figure 3: Variation of displacement, u vs. x *at* $t = 0.3$.

Figs. (1,2,3) shows the variation of temperature, stress and displacement with respect to distance, x under thermoelasticity model newly given by Quintanila [13] at time t=0.3. Fig. 1 indicates that when the boundary of the half space is subjected to a heat source varying exponentially with time and zero stress, the temperature decreases gradually and finally gets zero value after travelling a distance. Fig. 2 shows that first of all stress decreases from zero value then starts increasing with space variable x. Finally fig. 3 shows that displacement increases and after getting a maximum value starts decreasing with x.

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