On \((r^*g^*)^*\) Closed Sets in Topological Spaces

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Abstract: The aim of this paper is to introduce a new class of sets namely \((r^*g^*)^*\) closed sets in topological spaces. This class was obtained by generalizing closed sets via \(r^*g^*\) open sets which was introduced by N. Meenakumari and T. Indira [16]. This new class falls strictly between the class of closed sets and rg closed sets.

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1. Introduction

In 1970, Levine [12] introduced the concept of generalized closed set in the topological spaces. Many mathematicians started generalizing closed sets in recent years. In 1993, N. Palaniappan and K. Chandrasekhara Rao [22] introduced regular generalized closed (rg-closed) sets. M. K. R. S. Veerakumar introduced g* closed sets [26], g# closed sets [25] in topological spaces. The aim of this paper is to introduce a new class of sets namely \((r^*g^*)^*\) closed sets in topological spaces and study some basic properties.

2. Preliminaries

Definition 2.1: A subset \(A\) of a space \(X\) is called

1. a preopen set if \(A \subseteq \text{int}(\text{cl}(A))\) and a preclosed set if \(\text{cl}(\text{int}(A)) \subseteq A\).
2. a semi-open set if \(A \subseteq \text{cl}(\text{int}(A))\) and a semi-closed set if \(\text{int}(\text{cl}(A)) \subseteq A\).
3. an \(\alpha\)-open set if \(A \subseteq \text{int}(\text{cl}(\text{int}(A)))\) and a \(\alpha\)-closed set if \(\text{cl}(\text{int}(\text{cl}(A))) \subseteq A\).
4. A semi-preopen set (\(\beta\)-open) if \(A \subseteq \text{cl}(\text{int}(\text{cl}(A)))\) and a semi-preclosed set (\(\beta\)-closed) if \(\text{int}(\text{cl}(\text{int}(A))) \subseteq A\).

Definition 2.2: A subset \(A\) of a space \(X\) is called

1. A regular closed [24] if \(\text{cl}(\text{int}(A)) = A\).
2. A generalized closed (g-closed) [12] set if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open.
3. A regular generalized closed (rg-closed) [22] set if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open.
4. An \(\alpha\)-generalized closed (\(\alpha\)-g-closed) [8] set if \(\alpha \text{ cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open.
5. A semi generalized closed (briefly sg-closed) [6] if \(\text{sc}(A) \subseteq U\) whenever \((A) \subseteq U\) and \(U\) is semiopen in \(X\).
6. A generalized semi closed (briefly gs-closed) [2] if \(\text{sc}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).
7. A weekly generalized closed (briefly wg-closed) [13] if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).
8. A generalized pre regular closed (gpr closed) [10] if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open.
9. A \(g^*\) closed [26] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g^*\)-open.
10. A regular weakly generalized semi closed (rwg closed) [18] if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open.
11. A \(g^{**}\) closed [21] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g^{**}\)-open.
12. A \(g\#\) closed [25] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g\#\)-open.
13. A generalized semi-precloshed closed (gsp) closed [9] if \(\text{spcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open.
14. A generalized semi-precloshed star closed ((gsp)* closed) [20] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is gsp open.
15. A generalized precloshed closed (gp) closed [14] if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open.
16. A \(g^{*}\) closed [11] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g^{*}\)-open.
17. A regular\(^\alpha\) generalized closed (r\(^\alpha\)g-closed) [23] if \(\text{gcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open.
18. A regular generalized b-closed (rgb closed) [15] if \(\text{bcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open.
19. A mildly generalized closed (mildly g-closed) [17] if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g\)-open.
20. A \(r^*g^*\) closed set [16] if \(\text{rcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g\)-open.
3. Basic Properties of \((r*g^*)^*\) -Closed Sets

**Definition 3.1:** A subset \(A\) of a topological space \((X, \tau)\) is called a \((r*g^*)^*\)-closed set if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(r*g^*\)-open.

**Example 3.2:**
Let \(X = \{a, b, c\}\), \(\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}\)
Closed sets = \(\{\phi, X, \{b, c\}, \{a, c\}, \{c\}\}\)
\(r*g^*\)open sets of \(X\) are \(\phi, X, \{a\}, \{b\}, \{a, b\}\)
\((r*g^*)^*\) closed set are \(\{\phi, X, \{c\}, \{b, c\}, \{a, c\}\}\)

**Proposition 3.3**
Every closed set is \((r*g^*)^*\)-closed.

**Proof:**
Let \(A\) be any closed set in \(X\)
To prove : \(A\) is \((r*g^*)^*\)-closed set
Let \(A \subseteq U\) and \(U\) be any \(r*g^*\)-open set in \(X\)
\(A\) is closed \(\Rightarrow \text{cl}(A) = A \subseteq U\)
\(\Rightarrow \text{cl}(A) \subseteq U\)
\(\Rightarrow A\) is \((r*g^*)^*\)-closed in \(X\).
Hence every closed set is \((r*g^*)^*\)-closed.
The converse is not true as seen from the following example.

**Example 3.4**
Let \(X = \{a, b, c\}\), \(\tau = \{\phi, X \{a\}, \{a, c\}\}\).
Here \{a, b\} is \((r*g^*)^*\)closed but not \(r*g^*\)closed.

**Proposition 3.5**
Every \(g^*\) closed set is \((r*g^*)^*\) closed set.

**Proof:**
Let \(A \subseteq U\) Where \(U\) is \(r*g^*\)open.
Since every \(r*g^*\)open set is \(g^*\) open set we have \(\text{cl}(A) \subseteq U\). Therefore \(\text{cl}(A) \subseteq U\) where \(U\) is \(r*g^*\)open.Hence \(A\) is \((r*g^*)^*\)closed.
The converse is not true as seen from the following example.

**Example 3.6**
Let \(X = \{a, b, c\}\), \(\tau = \{\phi, X \{a\}, \{b, c\}\}\).
Here \{b\} is \((r*g^*)^*\) closed but not \(g^*\) closed.

**Proposition 3.7**
Every \((r*g^*)^*\) -closed set is \(rg\)-closed.

**Proof:**
Let \(A\) be any \((r*g^*)^*\) -closed set in \(X\)
To prove : \(A\) is \(rg\)-closed set
Let \(A \subseteq U\) and \(U\) be a regular-open set in \(X\).  
Since every regular open set is \(r*g^*\)open we have \(\text{cl}(A) \subseteq U\). Therefore \(\text{cl}(A) \subseteq U\) where \(U\) is \(r*g^*\)open.Hence \(A\) is \((r*g^*)^*\)closed.

**Example 3.8**
The converse need not be true as seen from the following example
Let \(X = \{a, b, c\}\) and \(\tau = \{X, \phi, \{a\}, \{a, b\}\}\)
Here \{a, b\} is \(rg\) closed but not \((r*g^*)^*\) closed.

**Proposition 3.9**
Every \((r*g^*)^*\) -closed set is \(gpr\) closed.

**Proof:**
Let \(A \subseteq U\) and \(U\) be an regular open set in \(X\). Since every regular open set is \(r*g^*\)open we have \(\text{cl}(A) \subseteq U\).
But \(\text{pcl}(A) \subseteq \text{cl}(A) \subseteq U\)
\(\Rightarrow \text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(X\) which implies \(A\) is \(gpr\) closed.
Hence every \((r*g^*)^*\) closed set is \(gpr\)-closed.

**Example 3.10**
The converse need not be true as seen from the following example.
Let \(X = \{a, b, c\}\) and \(\tau = \{X, \phi, \{a, b\}\}\)
Here \{a\} is \(gpr\) closed but not \((r*g^*)^*\) closed.

**Proposition 3.11**
Every \((r*g^*)^*\) -closed set is \(rwg\)-closed.

**Proof:**
Let \(A \subseteq U\) and \(U\) be an regular open . Since every regular open set is \(r*g^*\)open we have \(\text{cl}(A) \subseteq U\).
But \(\text{cl}(\text{int}(A)) \subseteq \text{cl}(A) \subseteq U\)
\(\Rightarrow \text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(X\).
Hence \(A\) is \(rwg\)-closed set in \(X\).
Hence every \((r*g^*)^*\) closed set is \(rwg\)-closed.

**Example 3.12**
The converse need not be true as seen from the following example.
Let \(X = \{a, b, c\}\), \(\tau = \{X, \phi, \{a\}\}\)
Here \{a\} is \(gpr\) closed but not \(rwg\) closed.

**Proposition 3.13**
Every \(g**\) closed set is \((r*g^*)^*\) closed set.

**Proof:**
Let \(A \subseteq U\) Where \(U\) is \(r*g^*\)open.
Since every \(r*g^*\)open set is \(g^*\) open set we have \(\text{cl}(A) \subseteq U\).
Hence \(\text{cl}(A) \subseteq U\) where \(U\) is \(r*g^*\)open.

**Example 3.14**
The converse need not be true as seen from the following example.
Let \(X = \{a, b, c\}\), \(\tau = \{X, \phi, \{a\}, \{a, c\}\}\)
Here \{a\} is \((r*g^*)^*\) closed but not \(g**\) closed.

**Proposition 3.15**
Every \(g^#\) closed set is \((r*g^*)^*\) closed set.

**Proof:**
Let \(A \subseteq U\) and \(U\) be an \(r*g^*\)open set in \(X\). Since every \(r*g^*\) open set is \(g^*\) open we have \(\text{cl}(A) \subseteq U\).
Hence \(\text{cl}(A) \subseteq U\) where \(U\) is \(r*g^*\)open.

**Example 3.16**
The converse need not be true as seen from the following example.

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Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b, c\}\} \)
Here \( \{b\} \) is \((r^g*)^*\) closed but not \(g^\#\) closed.

**Proposition 3.17:**
Every \((gsp)^*\) closed set is \((r^g*)^*\)-closed.

**Proof:**
Proof follows from the definition of \((gsp)^*\) closed set and fact that \(r^g*\) open implies \(gsp\) open.

**Example:** 3.18
The converse need not be true as seen from the following example
Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{b\}, \{a, c\}\}\)
Here \( \{a, c\} \) is not \((gsp)^*\) closed but \((r^g*)^*\) closed.

**Proposition 3.19:**
Every \(g^p^*\) closed set is \((r^g*)^*\)-closed.

**Proof:**
Proof follows from the definition of \((gp)^*\) closed set and fact that \(r^g*\) open implies \(gp\) open.

**Example:** 3.20
The converse need not be true as seen from the following example
\( X = \{a, b, c\}, \tau = \{X, \phi, \{c\}, \{a, c\}\}\)
Here \( \{a, c\} \) is not \((gp)^*\) closed but \((r^g*)^*\) closed.

**Proposition 3.21:**
Every \((r^g*)^*\) closed set is \(r^g\)-closed.

**Proof:**
Let \( A \) and \( U \) be an regular open set in \( X \). Since every regular open set is \(r^g*\) open we have \(cl(A) \subseteq U\)
But \(gcl(A) \subseteq cl(A) \subseteq U\)
\(\Rightarrow gcl(A) \subseteq U\) whenever \( A \subseteq U \) and \( U \) is regular open in \( X \)
Which implies \( A \) is \(r^g\) closed.
Hence every \((r^g*)^*\) closed set is \(r^g\)-closed

**Example:** 3.22
The converse need not be true as seen from the following example
\( X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, c\}\}\)
Here \( \{a, b\} \) is \((r^g)\) closed but not \((r^g*)^*\) closed.

**Proposition 3.23:**
Every \((r^g*)^*\) closed set is \(rgb\)-closed.

**Proof:**
Let \( A \) and \( U \) be an regular open set in \( X \). Since every regular open set is \(r^g*\) open we have \(cl(A) \subseteq U\)
But \(bcl(A) \subseteq cl(A) \subseteq U\)
\(\Rightarrow bcl(A) \subseteq U\) whenever \( A \subseteq U \) and \( U \) is regular open in \( X \)
Which implies \( A \) is \(rgb\) closed.

**Example:** 3.24
The converse need not be true as seen from the following example
Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{c\}\}\)
Here \( \{b\} \) is \(rgb\) closed but not \((r^g*)^*\) closed.
Thus We have the following diagram.

**Remark:** 3.25
\((r^g*)^*\) closed sets and \(semi\) closed sets are independent of each other as seen from the following examples.

**Example:** 3.26
Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b, a, b\}\}\). Here \( \{b\} \) is \(sem\) closed but not \((r^g*)^*\) closed

**Remark:** 3.27
\((r^g*)^*\) closed sets and \(semi\) \(pre\) closed sets are independent of each other as seen from the following examples.
Example 3.28
Let \( X=\{a,b,c\}, \tau =\{X, \phi, \{a\}, \{b\}, \{a,b\}\} \). Here \{a\} is semi pre closed but not \((r^g)^*\) closed.

Example 3.29: Let \( X=\{a,b,c\}, \tau =\{X, \phi, \{a\}, \{a,c\}\} \). Here \{b\} is \((r^g)^*\) closed but not semi pre closed.

Remark: 3.30
The following example shows that \((r^g)^*\) closedness is independent from sg closedness, gs closedness, pre closedness and wg closedness.

Example 3.31
Let \( X=\{a,b,c\}, \tau =\{X, \phi, \{a\}, \{a,c\}\} \)
(i) \{c\} is sg closed but not \((r^g)^*\) closed .\{a\} is \((r^g)^*\) closed but not sg closed .
(ii) \{c\} is gs closed but not \((r^g)^*\) closed .\{a\} is \((r^g)^*\) closed but not gs closed.
(iii) \{c\} is pre closed but not \((r^g)^*\) closed .\{a\} is \((r^g)^*\) closed but not pre closed .
(iv) \{c\} is wg closed but not \((r^g)^*\) closed .\{a\} is \((r^g)^*\) closed but not wg closed

Remark: 3.32
\((r^g)^*\) closed sets and \(\alpha\) -closed sets are independent of each other as seen from the following examples.

Example: 3.33
Let \( X=\{a,b,c\}, \tau =\{X, \phi, \{a\}, \{b\}, \{a,b\}\} \). Here \{c\} is \((r^g)^*\) closed but not \(\alpha\) -closed .
Let \( X=\{a,b,c\}, \tau =\{X, \phi, \{b\}, \{a,b\}\} \). Here \{a\} is \(\alpha\) -closed but not \((r^g)^*\) closed.

Thus we have the following diagram:

\[ A \not\subseteq B \text { represents } A \text { and } B \text { are independent of Each other. } \]

Theorem: 3.34
The union of two \((r^g)^*\)-closed sets is \((r^g)^*\)-closed set.
To prove: \(B\) is \((r^g)^*\)-closed set.

Let \(B \subseteq U\). Now \(cl(B) \subseteq cl(A) \subseteq U\)

\[\Rightarrow cl(B) \subseteq U\] whenever \(B \subseteq U\) and \(U\) is open.

\(\Rightarrow B\) is \((r^g)^*\)-closed

4. \((r^g)^*\) Open Sets

Definition: 4.1 A set \(A \subseteq X\) is called \((r^g)^*\) open set if its complement is \((r^g)^*\) closed.

Theorem: 4.2 A subset \(A \subseteq X\) is \((r^g)^*\) open iff there exists a \((r^g)^*\) closed set \(F\) such that \(F \subseteq \text{int} A\) whenever \(F \subseteq A\).

Proof: Let \(A\) be a \((r^g)^*\) closed and \(F \subseteq A\) then \(X-A \subseteq X-F\) where \(X-F\) is \((r^g)^*\) open. Therefore \(Cl(X-A) \subseteq X-F\) which implies \(X-(\text{int} A) \subseteq X-F\).

Conversely suppose \(F \subseteq \text{int} A\) where \(F\) is \((r^g)^*\) closed. \(F \subseteq A\).

Let \(X-A \subseteq U\) where is \(U\) is \((r^g)^*\) open.

Then \(X-U \subseteq A\) where \(X-U\) is \((r^g)^*\) closed by hypothesis \(X-U \subseteq \text{int} A\) which implies \(X-(\text{int} A) \subseteq U\).

Which implies \(Cl(X-A) \subseteq U\) where \(U\) is \((r^g)^*\) open and hence \(X-A\) is \((r^g)^*\) closed.

Hence \(A\) is \((r^g)^*\) open.

Theorem 4.3: If \(\text{int} A \subseteq B \subseteq A\) and \(A\) is \((r^g)^*\) open then \(B\) is \((r^g)^*\) open.

Proof: \(B \subseteq A\) implies \(X-A \subseteq X-B\).

Int \(A \subseteq B\) implies \(X-B \subseteq X-A\).

That is \(X-A \subseteq X-B \subseteq X-\text{int} A = Cl(X-A)\).

Since \(X-A\) is \((r^g)^*\) closed, by theorem (3.39) \(X-B\) is \((r^g)^*\) closed which implies \(B\) is \((r^g)^*\) open.

Theorem 4.5: Let \(B \subseteq X\) if \(B\) is \((r^g)^*\) open and \(int B \subseteq A\) then \(A \cap B\) is \((r^g)^*\) open.

Proof: It is given that \(int B \subseteq A\) and also \(int B \subseteq B\), implies \(Int B \subseteq A \cap B \subseteq B\).

By theorem 3.39 \(A \cap B\) is \((r^g)^*\) open.

Theorem 4.6: A set \(A\) is \((r^g)^*\) closed if \(cl(A) - A\) is \((r^g)^*\) open.

Proof: Let \(A\) be \((r^g)^*\) closed then \(F = \emptyset\), \(F \subseteq \text{int} (A)\).

Hence \(cl(A) - A\) is \((r^g)^*\) open by theorem (4.2).

5. Conclusion

In this paper we have introduced \((r^g)^*\)-closed and \((r^g)^*\)-open sets and studied some properties. This class of sets can be used to discuss the notion of Continuity, Compactness and connectedness and can be extended to other topological spaces like Fuzzy & Bitopological Spaces.

References


