An Interesting Generalization of Fibonacci & Lucas Sequence

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Abstract: ***In this paper, we consider the generalisation of classical Fibonacci sequence and Lucas sequence. We consider the sequence$\{H_n\}$defined by the recurrence relation $H_n = H_{n-1} + H_{n-2}$; for all $n \geq 2$, with $H_0 = 2m, H_1 = k + m$; where m, k are fixed integers. The initial conditions are the sum of'k' times the initial conditions of Fibonacci sequence and 'm' times the initial conditions of Lucas sequence. Using the technique of generating functions, we obtain the extended Binet formula for H_n. We obtain some fascinating properties for this sequence. We also establish some amusing identities for this sequence displaying the relation betweenH_n, Fibonacci sequence and Lucas sequence.***

Keywords: Fibonacci sequence; Lucas sequence; generating function; Generalized Fibonacci sequence

1. Introduction

Classical Fibonacci sequence can be generalized in many ways: either by preserving the recurrence relation and altering the initial conditions [1], [2], [5], [6]or by conserving the initial conditions of the sequence and modifying the recurrence relation[3], [4]. In this paper, we consider the generalized Fibonacci sequence$\{H_n\}$defined by the recurrence relation

$H_n = H_{n-1} + H_{n-2}$; for all $n \geq 2$, (1.1)

with $H_0 = 2m$ and $H_1 = k + m$; where m, k are fixed integers.We obtain extended Binet's formula using the techniques of generating functions and develop the connection formulae as well as negation formula for this sequence.

Here, the initial conditions H_0 and H_1 are the sum of'k' times the initial conditions of Fibonacci sequence and 'm' times the initial conditions of Lucas sequencerespectively. First few terms of sequence $\{H_n\}$ are: $2m, k+m, k+3m, 2k+4m, 3k+7m, \ldots$.

2. Extended Binet's Formula for H_n

It is well-known that the Binet formula for $\{F_n\}$ and $\{L_n\}$ are given by $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$ respectively. Here we obtain the extended Binet's formula for $\{H_n\}$.

Lemma 2.1: The generating function for the sequence$\{H_n\}$ is given by

$h(x) = \frac{2m+(k-m)x}{1-x-x^2}$.

Proof: Let $h(x) = H_0 + H_1 x + H_2 x^2 + \cdots$ be the generating function for the sequence $\{H_n\}$. Then we get $xh(x) = H_0 x + H_1 x^2 + H_2 x^3 + \cdots + H_n x^{n+1} + \cdots$and $x^2 h(x) = H_0 x^2 + H_1 x^3 + H_2 x^4 + \cdots + H_n x^{n+2} + \cdots$.
These gives

$$h(x) - xh(x) - x^2 h(x) = H_0 + (-H_0 + H_1)x + (-H_0 - H_1 + H_2)x^2 + \cdots$$

Using the definition of H_n, we get
$(1 - x - x^2)h(x) = 2m + (k - m)x$.
$\therefore h(x) = \sum_{n=0}^{\infty} H_n x^n = \frac{2m+(k-m)x}{1-ax-bx^2}$, which is the required generating function.

Remark: If we consider $1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$, then we have
$\alpha + \beta = 1, \alpha\beta = -1, \alpha^3 = 2\alpha + 1$. Also we see that $\alpha = \frac{1+\sqrt{5}}{2}$and $\beta = \frac{1-\sqrt{5}}{2}$.
We now obtain the extended Binet's formula for the sequence $\{H_n\}$.

Theorem 2.2:$H_n = k\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + m(\alpha^n + \beta^n)$.

Proof: If we consider $h(x) = \frac{2m+(k-m)x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$, then it can be seen that
$A = \frac{2m\alpha+(k-m)}{\alpha-\beta}$and $B = -\frac{2m\beta+(k-m)}{\alpha-\beta}$.

$$\text{Thus } h(x) = \frac{2m+(k-m)x}{1-x-x^2} = \frac{\frac{2m\alpha+(k-m)}{\alpha-\beta}}{1-\alpha x} + -\frac{\left(\frac{2m\beta+(k-m)}{\alpha-\beta}\right)}{1-\beta x}$$
$$= \frac{2m\alpha + (k-m)}{\alpha - \beta} \sum_{n=0}^{\infty} \alpha^n x^n - \left(\frac{2m\beta + (k-m)}{\alpha - \beta}\right) \sum_{n=0}^{\infty} \beta^n x^n$$
$$= 2m \sum_{n=0}^{\infty} \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) x^n + (k - m) \sum_{n=0}^{\infty} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) x^n.$$

This gives
$\sum_{n=0}^{\infty} H_n x^n = \left[2m \sum_{n=0}^{\infty} \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + (k - \right.$
m)n=0∞αn−βnα−βxn.

Thus

$$H_n = 2m\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + (k-m)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$
$$= k\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + m\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + \frac{m}{\alpha - \beta}(\alpha^{n+1} - \beta^{n+1} - \alpha^n + \beta^n)$$
$$= k\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + m\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + \frac{m}{\alpha - \beta}\left(-\frac{\alpha^n}{\beta} + \frac{\beta^n}{\alpha} + \frac{\alpha^{n-1}}{\beta} - \frac{\beta^{n-1}}{\alpha}\right)$$

$$= k\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)+m\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)+$$
$$\frac{m}{\alpha-\beta}\left(\frac{\alpha^{n-1}}{\beta}(1-\alpha)-\frac{\beta^{n-1}}{\alpha}(1-\beta)\right)$$
$$= k\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)+m\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)$$
$$+\frac{m}{\alpha-\beta}\left(\frac{\alpha^{n-1}}{\beta}\beta-\frac{\beta^{n-1}}{\alpha}\alpha\right)$$
$$= k\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)+\frac{m}{\alpha-\beta}(\alpha^{n+1}-\beta^{n+1}+\alpha^{n-1}-\beta^{n-1})$$
$$= k\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)+\frac{m}{\alpha-\beta}\left(\alpha^n\left(\alpha+\frac{1}{\alpha}\right)-\beta^n\left(\beta+\frac{1}{\beta}\right)\right)$$
$$= k\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)+\frac{m}{\alpha-\beta}(\alpha^n(\alpha-\beta)-\beta^n(\beta-\alpha))$$
$$= k\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)+\frac{m}{\alpha-\beta}((\alpha^n+\beta^n)(\alpha-\beta))$$
$= k\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)+m(\alpha^n+\beta^n)$, as required.

We use this result to express H_n as a linear combination of F_n and L_n.

Corollary 2.3: $H_n = kF_n + mL_n$.

Proof: Using the Binet's formula for the sequences $\{H_n\}$, Fibonacci Sequence $\{F_n\}$ and Lucas Sequence $\{L_n\}$, we get

$$H_n = k\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)+m(\alpha^n+\beta^n)$$
$$= kF_n + mL_n.$$

3. Some Summation Formulae for H_n

We first find the value of sum of first n terms of the sequence $\{H_n\}$.

Lemma 3.1: $H_1 + H_2 + \cdots + H_n = H_{n+2} - (3m+k)$.

Proof:

We have $H_n = H_{n-1} + H_{n-2} \Rightarrow H_{n-2} = H_n - H_{n-1}$.

Substituting $n = 3, 4, 5, \ldots$ successively, we get

$H_1 = H_3 - H_2, H_2 = H_4 - H_3, \ldots, H_n = H_{n+2} - H_{n+1}$.

Adding all these equations, we get

$H_1 + H_2 + \cdots + H_n = (H_3 + H_4 + \cdots + H_{n+2}) - (H_2 + H_3 + \cdots + H_{n+1})$.
$\therefore \sum_{i=1}^{n} H_i = H_{n+2} - H_2 = H_{n+2} - (3m+k)$, as required.

Lemma 3.2: $H_1 + H_3 + \cdots + H_{2n-1} = H_{2n} - 2m$.
Proof: We have $H_{n-1} = H_n - H_{n-2}$. Replacing n by $2, 4, 6, \ldots$ successively, we get
$H_1 = H_2 - H_0, H_3 = H_4 - H_2, \ldots, H_{2n-1} = H_{2n} - H_{2n-2}$.
Adding all these equations, we get
$H_1 + H_3 + \cdots + H_{2n-1} = (H_2 + H_4 + \cdots + H_{2n}) - (H_0 + H_2 + \cdots + H_{2n-2})$.
$\therefore \sum_{i=1}^{n} H_{2i-1} = H_{2n} - H_0 = H_{2n} - 2m$.

Lemma 3.3: $H_2 + H_4 + \cdots + H_{2n} = H_{2n+1} - (k+m)$.
This result can be proved easily using the earlier techniques or otherwise.

Lemma 3.4: $H_1 - H_2 + H_3 - H_4 + \cdots + (-1)^{n+1}H_n = (-1)^{n+1}H_{n-1} + k - m$.

Proof: From Lemma 3.2 and 3.3, we get
$$H_1 - H_2 + H_3 - H_4 + \cdots + H_{2n-1} - H_{2n} = H_{2n} - 2m - H_{2n+1} + (k+m)$$
$= -H_{2n-1} + k - m$.

Adding H_{2n+1} on both sides we get,
$$H_1 - H_2 + H_3 - H_4 + \cdots + H_{2n-1} - H_{2n} + H_{2n+1} = H_{2n+1} - H_{2n-1} + k - m$$
$= H_{2n} + k - m$.

Combining last two equations, we write
$H_1 - H_2 + H_3 - H_4 + \cdots + (-1)^{n+1}H_n = (-1)^{n+1}H_{n-1} + k - m$.

We next find the expression for the sum of squares of first n terms of sequence $\{H_n\}$.

Lemma 3.5: $H_1^2 + H_2^2 + H_3^2 + \cdots + H_n^2 = H_nH_{n+1} - 2m(k+m)$.
Proof: We have
$$H_nH_{n+1} = H_n(H_n + H_{n-1}) = H_n^2 + H_nH_{n-1} = H_n^2 + (H_{n-1} + H_{n-2})H_{n-1} = H_n^2 + H_{n-1}^2 + H_{n-2}H_{n-1}$$
Continuing in this way, we get
$$H_nH_{n+1} = H_n^2 + H_{n-1}^2 + \cdots + H_1^2 + H_1H_0$$
$\therefore H_nH_{n+1} = H_n^2 + H_{n-1}^2 + \cdots + H_1^2 + 2m(k+m)$. Thus, $H_1^2 + H_2^2 + H_3^2 + \cdots + H_n^2 = H_nH_{n+1} - 2m(k+m)$.

Corollary 3.6: $H_n^2 = H_nH_{n+1} - H_{n-1}H_n$; for $n \geq 1$.
Proof: From the proof of Lemma 3.5, we have $H_nH_{n+1} = H_n^2 + H_{n-1}^2 + \cdots + H_1^2 + H_1H_0$ and $H_{n-1}H_n = H_{n-1}^2 + \cdots + H_1^2 + H_1H_0$.
Subtracting we get $H_n^2 = H_nH_{n+1} - H_{n-1}H_n$; $n \geq 1$.

Lemma 3.7: $H_n^2 + H_{n+1}^2 = H_1H_{2n+1} + H_0H_{2n}$.
Proof: We prove the result by induction on n.
For $n = 1, H_1^2 + H_2^2 = (k+m)^2 + (k+3m)^2 = 2k^2 + 10m^2 + 8km$.
Also $H_1H_3 + H_0H_2 = (k+m)(2k+4m) + 2m(k+3m) = 2k^2 + 10m^2 + 8km$.

Thus result is true for $n = 1$. Assume that result is true for all integers n not exceeding some positive integer k. Thus $H_k^2 + H_{k+1}^2 = H_1H_{2k+1} + H_0H_{2k}$. Now,
$$H_{k+1}^2 + H_{k+2}^2 = H_{k+1}H_{k+1} + H_{k+2}H_{k+2}$$
$$= H_{k+1}(H_1F_{k+1} + F_kH_0) + H_{k+2}(H_1F_{k+2} + F_{k+1}H_0)$$
$$= H_1(H_{k+1}F_{k+1} + F_{k+2}H_{k+2}) + H_0(H_{k+1}F_k + F_{k+1}H_{k+2})$$
$$= H_1H_{2k+3} + H_0H_{2k+2}.$$

Thus result is true for $n = k+1$ also. This proves the result.

We use extended Binet formula for H_n to prove the following result.

Lemma 3.8: $H_3 + H_6 + H_9 + \cdots + H_{3n} = \frac{1}{2}(H_{3n+2} - (k + 3m)$.

Proof: By using extended Binet's formula from Theorem 2.2 we get

$$H_3 + H_6 + H_9 + \cdots + H_{3n} = k\left(\frac{\alpha^3-\beta^3}{\sqrt{5}}\right) + m(\alpha^3+\beta^3) + k\left(\frac{\alpha^6-\beta^6}{\sqrt{5}}\right) + m(\alpha^6+\beta^6) + \cdots$$

$$+k\left(\frac{\alpha^{3n}-\beta^{3n}}{\sqrt{5}}\right) + m(\alpha^{3n}+\beta^{3n})$$

$$= \frac{k}{\sqrt{5}}[(\alpha^3+\alpha^6+\cdots+\alpha^{3n}) - (\beta^3+\beta^6+\cdots+\beta^{3n})]$$

$$+m[(\alpha^3+\alpha^6+\cdots+\alpha^{3n}) - (\beta^3+\beta^6+\cdots+\beta^{3n})]$$

$$= \frac{k}{\sqrt{5}}\left[\left(\frac{\alpha^{3n+3}-\alpha^3}{\alpha^3-1}\right) - \left(\frac{\beta^{3n+3}+\beta^3}{\beta^3-1}\right)\right] + m\left[\left(\frac{\alpha^{3n+3}-\alpha^3}{\alpha^3-1}\right) - \left(\frac{\beta^{3n+3}+\beta^3}{\beta^3-1}\right)\right]$$

$$= \frac{k}{\sqrt{5}}\left[\left(\frac{\alpha^{3n+2}-\alpha^2}{2}\right) - \left(\frac{\beta^{3n+2}+\beta^2}{2}\right)\right] + m\left[\left(\frac{\alpha^{3n+2}-\alpha^2}{2}\right) + \left(\frac{\beta^{3n+2}+\beta^2}{2}\right)\right]$$

$$= \frac{1}{2}\left[\begin{array}{l}\left(k\left(\frac{\alpha^{3n+2}-\beta^{3n+2}}{\sqrt{5}}\right) + m(\alpha^{3n+2}+\beta^{3n+2})\right) \\ -\left(k\left(\frac{\alpha^{3n+2}-\beta^{3n+2}}{\sqrt{5}}\right) + m(\alpha^{3n+2}+\beta^{3n+2})\right)\end{array}\right]$$

$$= \frac{1}{2}[H_{3n+2} - H_2]$$

Thus $\sum_{i=1}^{n} H_{3i} = \frac{1}{2}[H_{3n+2} - (k+3m)]$, as required.

Lemma 3.9: $H_5 + H_8 + H_{11} + \cdots + H_{3n+2} = \frac{1}{2}(H_{3n+4} - (3k + 7m))$.

This result can be proved easily by the techniques used above.

We now derive the extended Cassini's identity which connects three consecutive H_n's together.

Lemma 3.10: $H_{n+1}H_{n-1} - H_n^2 = (-1)^{n+1}(5m^2 - k^2)$.

Proof: Here also we use the technique of mathematical induction over n.

It is easy to see that result is true for $n = 1$, for $H_2H_0 - H_1^2 = (k+3m)2m - (k+m)^2 = (-1)^2(5m^2 - k^2)$.

Now suppose that it is true for all positive integers not exceeding a positive integer 't'. Thus $H_{t+1}H_{t-1} - H_t^2 = (-1)^{t+1}(5m^2 - k^2)$ holds.

Now adding and subtracting H_tH_{t+1} onleft side we get,

$$H_{t+1}H_{t-1} - H_t^2 + H_tH_{t+1} - H_tH_{t+1} = (-1)^{t+1}(5m^2 - k^2)$$

$$\therefore H_{t+1}(H_t + H_{t-1}) - H_t^2 - H_tH_{t+1} = (-1)^{t+1}(5m^2 - k^2)$$

$$\therefore H_{t+1}^2 - H_t(H_t + H_{t+1}) = (-1)^{t+1}(5m^2 - k^2)$$

$$\therefore -(H_tH_{t+2} - H_{t+1}^2) = (-1)^{t+1}(5m^2 - k^2)$$

$\therefore H_tH_{t+2} - H_{t+1}^2 = (-1)^{t+2}(5m^2 - k^2)$, which is precisely our identity when $n = t + 1$. This proves the result by the induction.

4. Connection Formulae

Here we establish some amusing identities for the sequence $\{H_n\}$ displaying its relation with Fibonacci sequence and Lucas sequence.

Theorem 4.1:

(i) $H_{n+1} + H_{n-1} = (k+m)L_n + 2mL_{n-1}$ (ii) $H_{n+1} - H_{n-1} = (k+m)F_n + 2mF_{n-1}$;
where $n \geq 1$

Proof: We again use the induction technique to prove first result. Likewise second result can also be proved by induction.

For $n = 1$ we have $H_2 + H_0 = k + 5m$. Also $(k + m)L1+2mL0=k+5m$. Thus result is true for n=1.

We assume that the result is true for all integers not exceeding any positive integer 't'. Thus $H_{t+1} + H_{t-1} = (k+m)L_t + 2mL_{t-1}$ holds. Now,

$$H_{t+2} + H_t = H_{t+1} + H_t + H_{t-1} + H_{t-2}$$
$$= (H_{t+1} + H_{t-1}) + (H_t + H_{t-2})$$
$$= (k+m)L_t + 2mL_{t-1} + (k+m)L_{t-1} + 2mL_{t-2}$$
$$= (k+m)(L_t + L_{t-1}) + 2m(L_{t-1} + L_{t-2})$$
$$= (k+m)L_{t+1} + 2mL_t.$$

This proves the result for $n = t + 1$ and eventually for all positive integers n.

We next prove more generalized form of extended Catalan's identity which connects three consecutive H_n's with suffixes in arithmetic progression with F_n, for fixed n.

Theorem 4. $H_n^2 - H_{n+r}H_{n-r} = (-1)^{n-r-1}(5m^2 - k^2)F_r^2$.

Proof: Using theorem 2.2, we have

$$H_n^2 - H_{n+r}H_{n-r} = \left[k\left(\frac{\alpha^n-\beta^n}{\sqrt{5}}\right) + m(\alpha^n+\beta^n)\right]^2 - \left[k\left(\frac{\alpha^{n+r}-\beta^{n+r}}{\sqrt{5}}\right) + m(\alpha^{n+r}+\beta^{n+r})\right]$$

$$\times\left[k\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\sqrt{5}}\right) + m(\alpha^{n-r}+\beta^{n-r})\right]$$

$$= \frac{k^2}{5}\left(-2\alpha^n\beta^n + \alpha^n\beta^n\left(\frac{\beta}{\alpha}\right)^r + \alpha^n\beta^n\left(\frac{\alpha}{\beta}\right)^r\right) + m^2\left(\begin{array}{l}2\alpha^n\beta^n - \alpha^n\beta^n\left(\frac{\beta}{\alpha}\right)^r \\ -\alpha^n\beta^n\left(\frac{\alpha}{\beta}\right)^r\end{array}\right)$$

$$= \frac{k^2(\alpha\beta)^n}{5}\left[\frac{\alpha^{2r}+\beta^{2r}-2\alpha^r\beta^r}{(\alpha\beta)^r}\right] + m^2(\alpha\beta)^n\left[\frac{\alpha^{2r}+\beta^{2r}-2\alpha^r\beta^r}{(\alpha\beta)^r}\right]$$

$$= (\alpha\beta)^{n-r}(\alpha^r-\beta^r)^2\left[\frac{k^2}{5} - m^2\right]$$

$$= (-1)^{n-r}\frac{(\alpha^r-\beta^r)^2}{5}(k^2-5m^2)$$
$$= (-1)^{n-r-1}(5m^2-k^2)F_r^2.$$

We finally derive an interesting result which connects H_{-n} with Fibonacci sequence and Lucas sequence.

Lemma 4.3: $H_{-n} = (-1)^n(mL_n - kF_n)$.
Proof: We have $H_n = k\left(\frac{\alpha^n-\beta^n}{\sqrt{5}}\right) + m(\alpha^n+\beta^n)$

$$\therefore H_{-n} = k\left(\frac{\alpha^{-n}-\beta^{-n}}{\sqrt{5}}\right) + m(\alpha^{-n}+\beta^{-n})$$
$$= \frac{k}{\sqrt{5}}\left(\frac{\beta^n-\alpha^n}{(\alpha\beta)^n}\right) + m\left(\frac{\beta^n+\alpha^n}{(\alpha\beta)^n}\right)$$
$$= \frac{k}{\sqrt{5}}(-1)^{n+1}(\alpha^n-\beta^n) + m(-1)^n(\alpha^n+\beta^n).$$

Hence, $H_{-n} = (-1)^n(mL_n - kF_n)$.

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