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A New kind of Weighted Pál -Type Interpolation on Laguerre Polynomial

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Abstract: ***In this paper we have studied a new kind of weighted Pál-Type interpolation on the two sets of the nodes of Laguerre polynomial in which one set consists of the nodes of $L_n^k(x)$***

$$L_n^k(x) = \frac{x^{-\alpha} e^x}{n!} D^n\,[e^{-x} x^{n+\alpha}]$$

while the other are nodes of $L_n^{k-1}(x)$. We prove the existence, uniqueness and explicit representation of fundamental polynomials on infinite interval. Also we give the birkoff quadrature formula for this type of interpolation.

Keywords: lacunary interpolation, Pál - Type interpolation, Laguerre Polynomial, Birkoff, quadrature formula, MSC 2000: 41 A 05 65 D 32

1. Introduction

Balázs. J., Turán.P. [1] [2] [3] [17],Mathur P. and Datta S. [9] and many other authors [5][7][8][10][12][14][15][16] have discussed about interpolation problems when the values of the function and its consecutive derivatives are prescribed at the given set of the points. But in the theory of lacunary interpolation we study the interpolation processes arising from the data which involves non- consecutive derivatives. In most general case Birkoff [4] investigated a polynomial $f_m(x)$ of degree $\le (m_1 + m_2 + \cdots m_\mu - 1)$ such that for given $x_\mu (\mu = 1,2, \dots, n)$ we prescribe the value of some m_μ derivatives with given indicies not necessarily consecutive ones. In 1975 Pál [11] proved that when the function valued are prescribed on one set of n points and derivative values on other set of n-1 points, then there exist no unique polynomial of degree ≤ 2n-2, but prescribing function value at one more point not belonging to former set of n points there exists a unique polynomial of degree ≤ 2n-1. Lénárd M. [6] also investigated the Pál – type interpolation problem on the nodes of Laguerre abscissas. In Pál – type interpolation the derivative values are prescribed at the zeroes of $\omega'(x)$ while the function values are prescribed at the zeroes of $\omega_n(x) = (x - x_1) \dots (x - x_n)$. The aim of this paper is to consider a different type of weighted Pál – type interpolation in a unified way on the nodes of Laguerre polynomial. Laguerre polynomial $L_n^{(k)}(x)$ $(k > -1)$ has n distinct real roots in [0,∞) and we obtain the zeroes of $L_n^{(k)}(x)$ and $L_n^{(k-1)}(x)$ form the interscaled system of nodal points

(1.1) $0 \le \xi_0 < \xi_1^* < \xi_1 < \cdots < \xi_{n-1} < \xi_n^* < \xi_n < \infty$

In this paper we consider the problem if $\{\xi_i\}_{i=1}^n$ and $\{\xi_i^*\}_{i=1}^n$ be the two sets of interscaled nodal points on the interval [0,∞) then we seek to determine a polynomial $R_n(x)$ of minimal possible degree 2n+k satisfying the interpolatory conditions

(1.2) $R_n(\xi_i^*) = \alpha_i,\ (\omega R_n)'(\xi_i) = \alpha_i',\ for\ i = 1(1)n$

(1.3) $R_n^{(j)}(\xi_0) = \alpha_0^{(j)},\ j = 0,1, \dots, k$

where α_i, α_i' and $\alpha_0^{(j)}$ are arbitrary real numbers. If Laguerre polynomials $L_n^{(k)}(x)$ and $L_n^{(k-1)}(x)$ have zeroes $\{\xi_i\}_{i=1}^n$ and $\{\xi_i^*\}_{i=1}^n$ respectively and $\xi_0 = 0$ then the problem is not regular in general but it becomes regular with weight function $\omega(x) = e^{-x} x^{-k}$. We prove existence, uniqueness and explicit representation of fundamental polynomials in sec. 4 and Birkoff quadrature formulae for the integration of f(x) on [0,∞) with respect to the weight function $\omega(x) = e^{-x}$.in sec.5

2. Preliminaries

In this section we shall give some well-known results which are as follws: As we know that the Laguerre polynomial is a constant multiple of a confluent hypergeometric function so the differential equation is given by

(2.1) $xD^2 L_n^k(x) + (1 + k - x) D L_n^k(x) + n L_n^k(x) = 0$

(2.2) $L_n^{(k-1)'}(x) = -L_{n-1}^{(k)}(x)$

Also using the identities

(2.3) $L_n^{(k)}(x) = L_n^{(k+1)}(x) - L_{n-1}^{(k+1)}(x)$

(2.4) $x L_n^{(k)'}(x) = n L_n^{(k)}(x) - (n + k) L_{n-1}^{(k)}(x)$

We can easily find a relation

(2.5) $\frac{d}{dx}[x^k L_n^k(x)] = (n + k) x^{k-1} L_n^{(k-1)}(x)$

By the following conditions of orthogonality and normalization we define Laguerre polynomial $L_n^{(k)}(x), for\ k > -1$

(2.6) $\int_0^\infty e^{-x} x^k L_n^{(k)}(x) L_m^{(k)}(x) dx = \Gamma k + 1 \binom{n+k}{n} \delta_{nm}\ n, m = 0,1,2,\dots..$

(2.7) $L_n^{(k)}(x) = \sum_{\mu=0}^n \binom{n+k}{n-\mu} \frac{(-x)^\mu}{\mu!}$

The fundamental polynomials of Lagrange interpolation are given by

(2.8) $l_j(x) = \frac{L_n^{(k)}(x)}{L_n^{(k)'}(y_j)(x - y_j)} = \delta_{i,j}$

(2.9) $l_j^*(x) = \frac{L_n^{(k-1)}(x)}{L_n^{(k-1)'}(x_j)(x-x_j)} = \delta_{i,j}$

3. New Results

Theorem 1 : For n >1 fixed integer let $\{\alpha_i\}_{i=1}^n$, $\{\alpha_i^{'}\}_{i=1}^n$ and, $\{\alpha_0^{(j)}\}_{j=0}^k$ are arbitrary real numbers then there exists a unique polynomial $R_n(x)$ of minimal possible degree ≤ 2n+k on the nodal points (1.1) satisfying the condition (1.2) and (1.3). The polynomial $R_n(x)$ can be written in the form

(3.1) $\boldsymbol{R_n(x) = \sum_{j=1}^n A_j(x)\alpha_j + \sum_{j=1}^n B_j(x)\alpha_j^{'} +}$ ***j=0kCj(x)α0(j)***

Where $A_j(x)$, $B_j(x)$ and $C_j(x)$ are fundamental polynomials of degree ≤ 2n+k determined in lemma 1

Theorem 2 : Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$be the zeroes of Laguerre polynomials $L_n^{(k-1)}(x)$ and $L_n^{(k)}(x)$ respectively. then

(3.2) $\boldsymbol{\int_0^\infty f(x)e^{-x}dx \sim \sum_{j=1}^n A_j f(x_j) + \sum_{j=1}^n B_j f^{'}(y_j) +}$ ***j=0k−1Cjfj0***

Where A_j, B_j and C_j are the coefficients exist such that the quadrature formulae given above are exact for the polynomials of degree ≤ 2n+k

4. Proof of Theorem 1

To prove Theorem 1 we need to proof lemma 1.

Lemma 1 : For n > 1 the fundamental polynomials of the considered problem on the nodal points (1.1) is given by

(4.1) $A_j(x) = \frac{1}{x_j^{k+1}L_n^{(k)}(x_j)}[\, x^{k+1}l_j^*(x)L_n^{(k)}(x) - \frac{x^k L_n^{(k-1)}(x)}{L_n^{(k-1)'}(x_j)}\int_0^x \frac{tL_n^{(k)'}(t)+kL_n^{(k)}(t)}{t-x_j}dt]$

(4.2) $B_j(x) = \frac{e^{y_j}x^k L_n^{(k-1)}(x)}{L_n^{(k-1)}(y_j)}\int_0^x l_j(t)dt$

(4.3) $C_j(x) = p_j(x)x^j L_n^{(k)}(x)L_n^{(k-1)}(x) + x^k L_n^{(k-1)}(x)[c_j - \int_0^x \frac{L_n^{(k)'}(t)p_j(t)+q_j(t)L_n^{(k)}(t)}{t^{k-j}}dt]$

$, j = 0,1,\dots,k-1$

(4.4) $C_k(x) = \frac{1}{k!L_n^{((k-1)}(0)}x^k L_n^{(k-1)}(x)$

Where $A_j(x)$, $B_j(x)$ and $C_j(x)$ are fundamental polynomials of degree ≤ 2n+k. $p_j(x)$ and $q_j(x)$ are polynomials of degree at most k-j-1. c_j is defined in (4.16)

Proof:

Let $A_j(x)$, $B_j(x)$ and $C_j(x)$ are polynomials of degree ≤ 2n+k satisfying conditions (4.5), (4.6) and (4.7) respectively.

(4.5) $\begin{cases} A_j(x_i) = \delta_{i,j} \\ [e^{-x}x^{-k}A_j(x)]_{x=y_i}^{'} = 0, i = 1(1)n \text{ and } l = \\ A_j^{(l)}(0) = 0 \end{cases}$

$0,1,\dots,k$

(4.6) $\begin{cases} B_j(x_i) = 0 \\ [e^{-x}x^{-k}B_j(x)]_{x=y_i}^{'} = \delta_{i,j}, i = 1(1)n \text{ and } l = \\ B_j^{(l)}(0) = 0 \end{cases}$

$0,1,\dots,k$

(4.7) $\begin{cases} C_k(x_i) = 0 \\ [e^{-x}x^{-k}C_k(x)]_{x=y_i}^{'} = 0, i = 1(1)n \text{ and } l = \\ C_k^{(l)}(0) = \delta_{l,k} \end{cases}$

$0,1,\dots,k$

To determine $A_j(x)$ let

(4.8)

$A_j(x) = C_1[\, x^{k+1}l_j^*(x)L_n^{(k)}(x) + C_2 x^k L_n^{(k-1)}(x)\int_0^x \frac{tL_n^{(k)'}(t)+C_3L_n^{(k)}(t)}{t-x_j}dt]$

Where C_1, C_2 and C_3 are constants. $l_j^*(t)$ is defined in (2.9). As $A_j(x)$ is a polynomial of degree ≤ 2n+k so the integrand in (4.8) must be a polynomial of at most degree n which implies

(4.9) $tL_n^{(k)'}(t) + C_3L_n^{(k)}(t) = 0$

By using (2.3), (2.4) and (2.8) we determine

(4.10) $C_3 = k$, $C_1 = \frac{1}{x_j^{(k+1)}L_n^{(k)}(x_j)}$

Since $A_j(x)$ satisfies the conditions (4.5) by which we obtain

(4.11) $C_2 = -\frac{1}{x_j^{k+1}L_n^k(x_j)L_n^{(k-1)'}(x_j)}$

Hence we find the first fundamental polynomial $A_j(x)$of degree ≤ 2n+k

Again let

(4.12) $B_j(x) = C_5 x^k L_n^{(k-1)}(x)\int_0^x l_j(t)dt$

Where C_5 is a constant, $l_j(t)$ is defined in (2.8) and $B_j(x)$ is polynomial of degree ≤ 2n+k satisfying the conditions (4.6) by which we obtain

(4.13) $C_5 = \frac{e^{y_j}}{L_n^{(k-1)}(y_j)}$

Hence we find the second fundamental polynomial $B_j(x)$ of degree ≤ 2n+k

To determine $C_j(x)$ we consider $C_j(x)$ for fixed $j \in \{0,1,\dots\dots,k-1\}$ in the form

(4.14) $C_j(x) = p_j(x)x^j L_n^{(k)}(x)L_n^{(k-1)}(x) + x^k L_n^{(k-1)}(x)g_n(x)$

Where $p_j(x)$ and $g_n(x)$ are polynomials of degree k-j-1 and n respectively. Now it is obvious that $C_j^{(l)}(0) = 0$ for $(l = 0,\dots\dots,j-1)$ and since $L_n^{(k-1)}(x_i) = 0$ we get $C_j(x_i) = 0$ for $i = 1(1)n$. The coefficient of the polynomial $p_j(x)$ are calculated by the system

(4.15) $C_j^{(l)}(0) = \frac{d^l}{dx^l}[p_j(x)x^j L_n^{(k)}(x)L_n^{(k-1)}]_{x=0} = \delta_{i,j}$ $(l = j,\dots\dots,k-1)$

now from the equation $C_j^{(k)}(0) = 0$ we get

(4.16) $c_j = g_n(0) = \frac{-1}{k!L_n^{(k-1)}(0)}\frac{d^k}{dx^k}[p_j(x)x^j L_n^{(k)}(x)L_n^{(k-1)}(x)]_{x=0}$

Now using the condition $[e^{-x}x^{-k}C_j(x)]'_{x=y_i} = 0$ of (4.7), we get

(4.17) $g_n'(y_i) = -(y_i)^{j-k} L_n^{(k)'}(y_i) p_j(y_i)$

Which implies $g_n'(x)$ as follows

(4.18) $g_n'(x) = -\frac{L_n^{(k)'}(x)p_j(x)+q_j(x)L_n^{(k)}(x)}{x^{k-j}}$

Where $q_j(x)$ is a polynomial of degree k-j-1 and function $g_n'(x)$ will be a polynomial iff for $r = 0,1,\dots,k-j-1$

(4.19) $\frac{d^r}{dx^r}[L_n^{(k)'}(x)p_j(x) + q_j(x)L_n^{(k)}(x)]_{x=0} = 0$

The coefficients of $q_j(x)$ are uniquely calculated by this system. now integrating (4.18) we get

(4.20) $g_n(x) = g_n(0) + \int_0^x g_n'(t)dt$

Using (4.14) and (4.16) we obtain $C_j(x)$ of degree ≤ 2n+k satisfying the conditions (4.7).

Uniqueness and Existence : since $R_n(x)$ in (3.1) satisfies the conditions (1.2) and (1.3) hence the existence part is proved. now we seek to determine a polynomial $R_n(x)$ of minimal possible degree ≤ 2n+k satisfying the conditions (4.21) for $i = 1(1)n$ and $l = 0,1,\dots,k$

(4.21) $\begin{cases} S_n(x_i) = 0 \\ [e^{-x}x^{-k}S_n(x)]'_{x=y_i} = 0, i = 1(1)n \; and \; l = \\ S_n^l(0) = 0 \end{cases}$

$0,1,\dots,k$

Let us consider

(4.22) $S_n(x) = x^k L_n^{(k-1)}(x) s_n(x)$

Where $s_n(x)$ is a polynomial of at most degree n

(4.23) $[e^{-x}x^{-k}S_n(x)]'_{x=y_i} = e^{-y_i}L_n^{(k-1)}(y_i)s_n'(y_i) = 0$

By which $s_n'(y_i) = 0 \; for \; i = 1(1)n$ so that $s_n'(x) \equiv 0$ hence $s_n'(x) \equiv c$. so

(4.24) $S_n(x) = c\, x^k L_n^{(k-1)}(x)$, but

(4.25) $\frac{d^k S_n}{dx^k}(0) = c\, k!\, L_n^{(k-1)}(0) = 0$

Which inplies $c = 0$ as $L_n^{(k-1)}(0) \neq 0$ hence $S_n(x) \equiv 0$ which completes the proof of the uniqueness.

5. Proof of Theorem 2

Proof : Integrating (3.1) on [0,∞) with respect to the weight function e^{-x} we get

(5.1) $\int_0^\infty R_n(x)e^{-x}dx \sim \sum_{j=1}^n A_j f(x_j) + \sum_{j=1}^n B_j f'(y_j) +$ j=0nCjf(j)(0)

Where

(5.2) $A_j = \int_0^\infty A_j(x)e^{-x}dx \;\; j = 1(1)n$

(5.3) $B_j = \int_0^\infty B_j(x)e^{-x}dx \;\; j = 1(1)n$

(5.4) $C_j = \int_0^\infty C_j(x)e^{-x}dx \; j = 1(1)n$

Where $A_j(x)$, $B_j(x)$ and $C_j(x)$ are defined in theorem 1 hence the quadrature formulae given in theorem 2 are exact for the polynomials of degree ≤ 2n+k. now by the orthogonality.

(5.5)

$C_k = \int_0^\infty C_k(x)e^{-x}dx = \frac{1}{k!L_n^{(k-1)}(0)}\int_0^\infty L_n^{(k-1)}(x)x^k e^{-x}dx =$

0, for n >1

Now to prove Theorem 2 we need lemma 1

Lemma 2 : for $k \geq 1$ fixed integer the coefficients of the quadrature formula given in Theorem 2 are

(5.6) $A_j = \frac{(n+k)![x_j-(n+k-1)]}{x_j^{k+1}[L_n^{(k)}(x_j)]^2(n-1)n!}$

(5.7) $B_j = \frac{e^{y_j}y_j[y_j-(n+k)](n+k-1)!}{[L_n^{(k-1)}(y_j)]^2(n+k)(n-1)n!}$ for j=1(1)n

Proof : Let

(5.8) $\frac{L_n^{(k-1)}(x)}{x-x_j} = a_{j,n-1}x^{n-1} + a_{j,n-2}x^{n-2} + a_{j,n-3}x^{n-3} + \dots + a_{j,o}$

(5.9) $L_n^{(k-1)}(x) = (x-x_j)(a_{j,n-1}x^{n-1} + a_{j,n-2}x^{n-2} + a_{j,n-3}x^{n-3} + \dots + a_{j,o})$

Using (2.7) and (5.9) we get

(5.10) $a_{j,n-1} = \frac{(-1)^n}{n!}$, $a_{j,n-2} = \frac{(-1)^n}{n!}[x_j - n(n+k-1)]$

Now let

(5.11) $x\frac{L_n^{(k-1)}(x)}{(x-x_j)} = \sum_{i=0}^n C_{j,i}L_i^{(k)}(x)$

comparing the coefficients in (5.11) and using (2.7) we get

(5.12) $C_{j,n} = 1$, $C_{j,n-1} = -\frac{1}{n}[x_j + n]$

Now

(5.13) $\frac{1}{L_n^{(k-1)'}(x_j)}\int_0^\infty \frac{xL_n^{(k-1)}(x)}{(x-x_j)}x^k L_n^{(k)}(x)e^{-x}dx =$

Cj,n 1Ln(k−1)′(xj)0∞[Lnk(x)]2xke−xdx

Using (2.6), (5.11) and (5.13) we get

(5.14)

$\frac{1}{L_n^{(k-1)'}(x_j)}\int_0^\infty \frac{xL_n^{(k-1)}(x)}{(x-x_j)}x^k L_n^{(k)}(x)e^{-x}dx = \frac{1}{L_n^{(k-1)'}(x_j)}k!\binom{n+k}{n}$

$= \frac{(n+k)!}{L_n^{(k-1)'}(x_j)n!}$

In the similar way we can find

(5.15) $\int_0^\infty [x\int_0^x \frac{tL_n^{(k)'}(t)+kL_n^{(k)}(t)}{(t-x_j)}dt]L_n^{(k-1)}(x)x^{(k-1)}e^{-x}dx =$

xj−kn+kn−1(k−1)!n+k−1n

$= \frac{(x_j-k)(n+k)!}{(n-1)\,n!}$

Using (4.1), (5.14) and (5.15) we get

(5.16) $A_j = \int_0^\infty A_j(x)e^{-x}dx = -\frac{(n+k)![x_j-(n+k-1)]}{x_j^{k+1}L_n^{(k)}(x_j)L_n^{(k-1)'}(x_j)(n-1)n!}$

Using $L_n^{(k-1)'}(x_j) = -L_n^{(k)}(x_j)$ we get

(5.17) $A_j = \frac{(n+k)![x_j-(n+k-1)]}{x_j^{k+1}[L_n^{(k)}(x_j)]^2(n-1)n!}$

Now in the similar way we find

(5.18)

$B_j =$

$\int_0^\infty B_j(x)e^{-x}dx =$

eyj[yj−n+k]Lnk′yjLnk−1yj(n−1)0∞[Lnk−1x]2xk−1e−xdx

$$= \frac{e^{y_j}[y_j-(n+k)]}{L_n^{(k)'}(y_j)L_n^{(k-1)}(y_j)(n-1)}(k-1)!\binom{n+k-1}{n}$$

$$= \frac{e^{y_j}[y_j-(n+k)](n+k-1)!}{L_n^{(k)'}(y_j)L_n^{(k-1)}(y_j)(n-1)n!}$$

Now using

(5.19) $y_j L_n^{(k)'}(y_j) = (n+k)L_n^{(k-1)}(y_j)$

We get

(5.20) $B_j = \frac{e^{y_j} y_j [y_j-(n+k)](n+k-1)!}{[L_n^{(k-1)}(y_j)]^2(n+k)(n-1)n!}$

Which completes the proof.

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