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Common Fixed Point Theorem for Eight Mappings in Menger Space Using Rational Inequality Without Continuity

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Abstract: ***In this paper we establish Common fixed point theorem for eight mappings in Menger space using the notion of compatibility including rational term without continuity***

Keywords: Menger space, Weak compatibility, Common fixed point .

1. Introduction

There have been number of generalizations of metric spaces. One such generalization is Menger space introduced in 1942 by Menger [4] who used distribution functions instead of nonnegative real numbers as values of the metric. This space was expended rapidly with the pioneering of Schweizer and Sklar [5,6]. This concept plays a vital role in probabilistic functional analysis , nonlinear analysis and applications [1]. In 1972 , V.M. Sehgal and A.T. Bharucha-Reid [7] obtained a generalization of Banach contraction principle on a complete Menger space which is mile stone in development fixed point theory in Menger space. In 1986, jungck [2] introduced the notion of compatible mappings in metric spaces, And this condition has further been weakened by introducing the notion of weakly compatible mappings by Jungck and Rhoades [3]
The concept of weakly compatible mappings is most general as each pair of compatible mappings is most compatible but the reverse is not true. Recently Singh and jain [8] established a common fixed point theorem in Menger space using the concept of weak compatibility and compatibility of pair of self maps. In this paper we generalize and extend the result of Singh and jain [9] for eight mapping opposed to six mappings in complete Menger space using the concept of compatibility including rational term without continuity.

2. Preliminaries

Definition 2.1 A probabilistic metric space (PM-space) is an ordered pair (X,F) consisting of a non empty set X and a function F: X × X →L, where L is the collection of all distribution functions and the value of F at (u,v) ϵ X × X is represented by $F_{u,v}$. The function $F_{u,v}$ is assumed to satisfy the following conditions:

(PM- 1) $F_{u,v}(x) = 1$,for all x>0 if and only if u = v
(PM- 2) $F_{u,v}(0) = 0$;
(PM- 3) $F_{u,v} = F_{v,u}$;
(PM- 4) $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x+y) = 1$ for all u,v,w in X and $x, y > 0$.

Definition 2.2: A mapping t:[0, 1] x [0, 1] →[0, 1] is called a t- norm if

(a) t(a, 1) = a , t(0, 0) =0
(b) t(a,b) = t(b,a) (symmetric property)
(c) t(c,d) = t(a,b) for c ≥ a , d ≥ b
(d) t(t(a, b), c) = t(a, t(b, c))

Definition 2.3: A Menger space is a triplet (X, F, t) where (X, F) is a PM- space and t is a t-norm such that the inequqlity.

$F_{u,w}(x+y) \geq t\{ F_{u,v}(x) , F_{v,w}(y) \}$ for all u,v,w in X and $x, y > 0$

Definition 2.4: Self –maps A and S of a Menger space (X, F, t) are said to be weak compatible if they commute at their coincidence points i.e. if Ap= Sp for some $p \epsilon$ N then ASp=SAp

Lemma (1) : Let $\{x_n\}$ be a sequence in a Menger space (X, F, t) with continuous t-norm and t(x, x) ≥ x . suppose for all $x \epsilon [0, 1]$ there exists $k \epsilon (0, 1)$ such that for all x>0 and $n \epsilon$ N

$$F x_n, x_{n+1}(kx) \geq F x_{n-1}, x_n(x).$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma (2) : Let (X, F, t) be a Menger space . If there exists $k \epsilon (0, 1)$ such that for p,q ϵ X

$$F_{p,q}(kx) \geq F_{p,q}(x). \text{ Then } p=q$$

In 2006 , Bijendra Singh and shishir jain [9] introduced fixed point theorems in Menger space through semi-compatibility and gave the following fixed point theorem for six mappings:

Theorem: Let A, B, S, T, L and M are self mappings on a complete Menger space (X, F, min) satisfying:

(a) $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$.
(b) $AB = BA, ST = TS, LB = BL, MT = TM$
(c) Either AB or L is continuous.
(d) (L, AB) is semi-compatible and (M, ST) is weak –compatible.
(e) There exists $k \epsilon$ (0, 1) such that

$F_{Lp,Mq}$ (Kx) $\geq$ min{$F_{ABp,Lp}$ (x), $F_{STq,Mq}$ (x), $F_{STq,Lp}$ (βx), $F_{ABp,Mq}$ ((2 − β)x), $F_{ABp,STq}$ (x)}

For all p, qϵX,>0 and β(0,2). Then A, B, S, T, L, M have a unique common fixed point in X.

Now we are prove the following result.

3. Main Results

Theorem (3.1): Let A, B, S, T, L, M, P and Q are self mappings on a complete Menger space (X, F, t) satisfying:

(3.1.1) $A(X) \subseteq ST(X) \cap L(X) \cap M(X), B(X) \subseteq PQ(X)$.

(3.1.2) $PQ = QP, ST = TS, AQ = QA, BT = TB, LT = TL, MT = TM$.

(3.1.3) $(A, PQ), (L, ST), (B, ST), (L. M)$ are weak compatible.

(3.1.4) There exists $k\epsilon$ (0, 1) such that

$F_{Ax,By}$ (Kt) $\geq$ min{$F_{PQx,Ly}$ (t) $F_{STy,Ly}$ (αt), $F_{By,PQx}$ ((2 − α)t), $\left(\frac{F_{PQx,Ly}(t) \cdot F_{Ax,PQx}(t)}{F_{PQx,STy}(t)}\right), \left(\frac{F_{STy,Ax}(t) \cdot F_{Ax,My}(t)}{F_{Ax,Ly}(t)}\right)$}

For all $x, y \epsilon X, \alpha \epsilon (0,2) and\ t > 0$.

(3.1.5). If one of A(X), ST(X), PQ(X), B(X) is a complete sub space of X, then :

(a) A and PQ have a coincidence point.

(b) B and ST , L , M have a coincidence point.

Then self –maps A,B,S,T,L,M,P and Q have a unique common fixed point in X.

Proof :

Let x_0 ϵ X By condition 3.1.1 there exists x_1 , x_2 ϵ X such that

$Ax_0 = STx_1 = Lx_1 = Mx_1 = y_0$ and $Bx_1 = PQx_2 = y_1$. Inductively we can construct sequence { x_n } and { y_n } in X. such that $y_{2n} = Ax_{2n} = STx_{2n+1} = Lx_{2n+1} = Mx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = PQ\,x_{2n+2}$ for n = 0,1,2, …

putting x = x_{2n} and y = x_{2n+1} for x > 0 in 3.1.4 then we have

$$F_{Ax_{2n},Bx_{2n+1}}(Kt) \geq \min\{F_{PQx_{2n},Lx_{2n+1}}(t), F_{STx_{2n+1},Lx_{2n+1}}(\alpha t), F_{Bx_{2n+1},PQx_{2n}}((2-\alpha)t), \left(\frac{F_{PQx_{2n},Lx_{2n+1}}(t) \cdot F_{Ax_{2n},PQx_{2n}}(t)}{F_{PQx_{2n},STx_{2n+1}}(t)}\right), \left(\frac{F_{STx_{2n+1},Ax_{2n}}(t) \cdot F_{Ax_{2n},Mx_{2n+1}}(t)}{F_{Ax_{2n},Lx_{2n+1}}(t)}\right)\}.$$

$$F_{y_{2n},y_{2n+1}}(Kt) \geq \min\{F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n}}(\alpha t), F_{y_{2n+1},y_{2n-1}}((2-\alpha)t), \left(\frac{F_{y_{2n-1},y_{2n}}(t) \cdot F_{y_{2n},y_{2n-1}}(t)}{F_{y_{2n-1},y_{2n}}(t)}\right), \left(\frac{F_{y_{2n},y_{2n}}(t) \cdot F_{y_{2n},y_{2n}}(t)}{F_{y_{2n},y_{2n}}(t)}\right)\}.$$

Hence

$F_{y_{2n},y_{2n+1}}$ (Kt) $\geq$ min{ $F_{y_{2n-1},y_{2n}}$ (t), $F_{y_{2n},y_{2n}}$ (αt), $F_{y_{2n+1},y_{2n-1}}$ ((2 - α)t)}

Let β ϵ (0, 1) and put β =1 – α we get

$F_{y_{2n},y_{2n+1}}$ (Kt) $\geq$ min{ $F_{y_{2n-1},y_{2n}}$ (t), $F_{y_{2n},y_{2n}}$ ((1 – β)t), $F_{y_{2n+1},y_{2n-1}}$ ((1+β)t)}

$F_{y_{2n},y_{2n+1}}$ (Kt) $\geq$ min{ $F_{y_{2n-1},y_{2n}}$ (t), $F_{y_{2n-1},y_{2n}}$ (t), $F_{y_{2n},y_{2n+1}}$ (βt)}

$F_{y_{2n},y_{2n+1}}$ (Kt) $\geq$ min{ $F_{y_{2n-1},y_{2n}}$ (t), $F_{y_{2n},y_{2n+1}}$ (βt)}

Making $\beta \to 1$, we get

$F_{y_{2n},y_{2n+1}}$ (Kt) $\geq$ min{ $F_{y_{2n-1},y_{2n}}$ (t), $F_{y_{2n},y_{2n+1}}$ (t)}

Similarly, $F_{y_{2n+1},y_{2n+2}}$ (Kt) $\geq$ min{ $F_{y_{2n},y_{2n+1}}$ (t), $F_{y_{2n+1},y_{2n+2}}$ (t)}

Therefore for all n even or odd we have

$F_{y_n,y_{n+1}}$ (Kt) $\geq$ min{ F_{y_{n-1},y_n} (t), $F_{y_n,y_{n+1}}$ (t)}

Consequently, it follows that for p = 1,2,3, …

$F_{y_n,y_{n+1}}$ (Kt) $\geq$ min{ F_{y_{n-1},y_n} (t), $F_{y_n,y_{n+1}}$ $(\frac{t}{k^p})$} .

By noting that $F_{y_n,y_{n+1}}$ $(\frac{t}{k^p}) \to 1$ as $n \to \infty$ it follows that

$F_{y_n,y_{n+1}}$ (Kt) $\geq$ min{ F_{y_{n-1},y_n} (t) } for all $n\epsilon$ N and t > 0.

Hence by Lemma (1) , { y_n } is a Cauchy sequence in X. Now suppose PQ (X) is complete. Note that the sub sequence { y_{2n+1} } is contained in PQ (X) call it z. Let $u\ \epsilon PQ^{-1}$ (z) then PQu = z. we shall use the fact that subsequence { y_{2n} } also converges to z.

$Bx_{2n+1} \to z$, $STx_{2n+1} \to z$, $Lx_{2n+1} \to z$, $Mx_{2n+1} \to z$

$Ax_{2n} \to z$, PQ $x_{2n+2} \to z$. By putting x = u and y = x_{2n+1} with α = 1 in 3.1.4

$$F_{Au,Bx_{2n+1}}(Kt) \geq \min\{F_{PQu,Lx_{2n+1}}(t), F_{STx_{2n+1},Lx_{2n+1}}(t),, F_{Bx_{2n+1},PQu}(t), \left(\frac{F_{PQu,Lx_{2n+1}}(t) \cdot F_{Au,PQu}(t)}{F_{PQu,STx_{2n+1}}(t)}\right), \left(\frac{F_{STx_{2n+1},Au}(t) \cdot F_{Au,Mx_{2n+1}}(t)}{F_{Ax_{2n},Lx_{2n+1}}(t)}\right)\}$$

Taking lim $n \to \infty$we get

$$F_{Au,z}(Kt) \geq \min\{F_{z,z}(t), F_{z,z}(t), F_{z,z}(t), \left(\frac{F_{z,z}(t) \cdot F_{Au,z}(t)}{F_{z,z}(t)}\right), \left(\frac{F_{z,Au}(t) \cdot F_{Au,z}(t)}{F_{Au,z}(t)}\right)\}.$$

Thus we have

$F_{Au,z}$ (Kt) $\geq$ $F_{Au,z}$ (t)

Therefore by Lemma 2 we have $Au = z$. since $PQu = z$ thus we have $Au = PQu = z$ that is u is coincident point of A and PQ this proves 3.1.5 (a).

Since $A(X) \subseteq ST(X) \cap L(X) \cap M(X), Au = z$ implies that $ST(X) \cap L(X) \cap M(X)$. Then $STv = Lv = Mv = z$. By putting x = x_{2n+2} and y = v with α = 1 in 3.1.4

$F_{Ax_{2n+2},Bv}(Kt) \geq \min\{ F_{PQx_{2n+2},Lv}(t), F_{STv,Lv}(\alpha t), F_{Bv,PQx_{2n+2}}((2-\alpha)t), \left(\frac{F_{PQx_{2n+2},Lv}(t) \cdot F_{Ax_{2n+2},PQx_{2n+2}}(t)}{F_{PQx_{2n+2},STv}(t)}\right), \left(\frac{F_{STv,Ax_{2n+2}}(t) \cdot F_{Ax_{2n+2},Mv}(t)}{F_{Ax_{2n+2},Lv}(t)}\right)\}.$

Taking lim $n \to \infty$ we get

$F_{z,Bv}(Kt) \geq \min\{ F_{z,z}(t), F_{z,z}(t), F_{Bv,z}(t), \left(\frac{F_{z,z}(t) \cdot F_{z,z}(t)}{F_{z,z}(t)}\right), \left(\frac{F_{z,z}(t) \cdot F_{z,z}(t)}{F_{z,z}(t)}\right)\}.$

Thus we have

$F_{z,Bv}(Kt) \geq F_{z,Bv}(t)$. Therefore by Lemma (2) we have $Bv = z$ since $STv = Lv = Mv = z$ thus we have $Bv = STv = Lv = Mv = z$ that is v is coincident point of B and ST, L,M This proves (b).

The remaining two cases pertain essentially to the previous cases . Indeed if A(X) or B(X) is complete then by 3.1.4 $z \in A(X) \subset ST(X) \cap L(X) \cap M(X)$ or $z \in B(X) \subseteq PQ(X)$. Thus 3.1.5 (a) and (b) are completely established.
Since the pair {A,PQ } is weakly compatible therefore A and PQ commute at their coincidence point that is A(PQu) = (PQ)Au or Az = PQz.
Since the pair {B, ST} , {L, ST} and {L, M} are weakly compatible therefore
B(STv) = ST(Bv) or Bz = STz
L(STv) = ST(Lv) or Lz = STz
L(Mv) = M(Lv) or Lz = Mz Therefore Bz = STz =Lz =Mz.
By putting x= z, y = x_{2n+1} with α = 1 in 3.1.4

$F_{Az,Bx_{2n+1}}(Kt) \geq \min\{ F_{PQz,Lx_{2n+1}}(t), F_{STx_{2n+1},Lx_{2n+1}}(t), F_{Bx_{2n+1},PQz}(t), \left(\frac{F_{PQz,Lx_{2n+1}}(t) \cdot F_{Az,PQz}(t)}{F_{PQz,STx_{2n+1}}(t)}\right), \left(\frac{F_{STx_{2n+1},Az}(t) \cdot F_{Az,Mx_{2n+1}}(t)}{F_{Az,Lx_{2n+1}}(t)}\right)\}.$

Taking lim $n \to \infty$ we get

$F_{Az,z}(Kt) \geq \min\{ F_{z,z}(t), F_{z,z}(t), F_{z,z}(t), \left(\frac{F_{z,z}(t) \cdot F_{Az,z}(t)}{F_{z,z}(t)}\right), \left(\frac{F_{z,Az}(t) \cdot F_{Az,z}(t)}{F_{Az,z}(t)}\right)\}.$

Thus we have

$F_{Az,z}(Kt) \geq F_{Az,z}(t)$

Therefore by Lemma 2 we have Az = z so Az = PQz = z.
By putting x = x_{2n+2} and y = z with α = 1 in 3.1.4

$F_{Ax_{2n+2},Bz}(Kt) \geq \min\{ F_{PQx_{2n+2},Lz}(t), F_{STz,Lz}(t),, F_{Bz,PQx_{2n+2}}(t), \left(\frac{F_{PQx_{2n+2},Lz}(t) \cdot F_{Ax_{2n+2},PQx_{2n+2}}(t)}{F_{PQx_{2n+2},STz}(t)}\right), \left(\frac{F_{STz,Ax_{2n+2}}(t) \cdot F_{Ax_{2n+2},Mz}(t)}{F_{Ax_{2n+2},Lz}(t)}\right)\}.$

Taking lim $n \to \infty$ we get

$F_{z,Bz}(Kt) \geq \min\{ F_{z,Bz}(t), F_{Bz,Bz}(t), F_{Bz,z}(t), \left(\frac{F_{z,Bz}(t) \cdot F_{z,z}(t)}{F_{z,Bz}(t)}\right), \left(\frac{F_{Bz,z}(t) \cdot F_{z,Bz}(t)}{F_{z,Bz}(t)}\right)\}.$

Thus we have

$F_{z,Bz}(Kt) \geq F_{z,Bz}(t)$. Therefore by Lemma (2) we have Bz = z so Bz = STz = Lz = Mz = z
By putting x = z and y = Tz with α = 1 in 3.1.4

$F_{Az,BTz}(Kt) \geq \min\{ F_{PQz,LTz}(t), F_{STTz,LTz}(t), F_{BTz,PQz}(t), \left(\frac{F_{PQz,LTz}(t) \cdot F_{Az,PQz}(t)}{F_{PQz,STTz}(t)}\right), \left(\frac{F_{STTz,Az}(t) \cdot F_{Az,MTz}(t)}{F_{Az,LTz}(t)}\right)\}.$

As BT = TB, LT=TL,ST=TS,MT=TM
BTz=TBz=Tz, LTz=TLz=Tz, STTz=TSTz=Tz, MTz=TMz=Tz,
Taking lim $n \to \infty$ we get

$F_{z,Tz}(Kt) \geq \min\{ F_{z,Tz}(t), F_{Tz,Tz}(t), F_{Tz,z}(t), \left(\frac{F_{z,Tz}(t) \cdot F_{z,z}(t)}{F_{z,Tz}(t)}\right), \left(\frac{F_{Tz,z}(t) \cdot F_{z,Tz}(t)}{F_{zT,z}(t)}\right)\}.$

Thus we have

$F_{z,Tz}(Kt) \geq F_{z,Tz}(t)$

Therefore by Lemma 2 we have Tz = z, Since STz=z, therefore Sz=z
By putting x = Qz and y = z with α = 1 in 3.1.4

$F_{AQz,Bz}(Kt) \geq \min\{ F_{PQQz,Lz}(t), F_{STz,Lz}(t), F_{Bz,PQQz}(t), \left(\frac{F_{PQQz,Lz}(t) \cdot F_{AQz,PQQz}(t)}{F_{PQQz,STz}(t)}\right), \left(\frac{F_{STz,AQz}(t) \cdot F_{AQz,Mz}(t)}{F_{AQz,Lz}(t)}\right)\}.$

As AQ=QA, PQ=QP we have
AQz=QAz=Qz, PQQz=QPQz=Qz
Taking lim $n \to \infty$ we get

$F_{Qz,z}(Kt) \geq \min\{F_{Qz,z}(t), F_{z,z}(t), F_{z,Qz}(t), \left(\frac{F_{Qz,z}(t) \cdot F_{Qz,Qz}(t)}{F_{Qz,z}(t)}\right), \left(\frac{F_{z,Qz}(t) \cdot F_{Qz,z}(t)}{F_{Qz,z}(t)}\right)\}.$

Thus we have

$F_{Qz,Bz}(Kt) \geq F_{Qz,z}(t)$. Therefore by Lemma (2) we have Qz = z. Since PQz = z therefore Pz = z. By combining the above results we have Az = Bz = Lz = Mz = Sz = Tz = Pz = Qz = z. That is z is a common fixed point of A, B, L, M, S, T, P and Q.

Uniqueness: Let z' (z ≠ z') be another common fixed point of A, B, L, M, S, T, P and Q, then Az' = Bz' = Lz' = Mz' = Sz' = Tz' = Pz' = Qz' = z'.
By putting x = z and y = z' with α = 1 in 3.1.4 we have

$F_{Az,Bz'}(Kt) \geq \min\{ F_{PQz,Lz'}(t), F_{STz',Lz'}(t), F_{Bz',PQz}(t), \left(\frac{F_{PQz,Lz'}(t) \cdot F_{Az,PQz}(t)}{F_{PQz,STz'}(t)}\right), \left(\frac{F_{STz',Az}(t) \cdot F_{Az,Mz'}(t)}{F_{Az,Lz'}(t)}\right) \}.$

$F_{z,z'}(Kt) \geq \min\{ F_{z,z'}(t), F_{z',z'}(t), F_{z',z}(t), \left(\frac{F_{z,z'}(t) \cdot F_{z,z}(t)}{F_{z,z'}(t)}\right), \left(\frac{F_{z',z}(t) \cdot F_{z,z'}(t)}{F_{z,z'}(t)}\right) \}.$

$F_{z,z'}(Kt) \geq F_{z,z'}(t)$. by Lemma (2) z= z'. Therefore z is a common fixed point of A, B, L, M, S, T, P and Q. This completes the proof.

Corollary 3.1.6 : Let A,S, T, L, M, P and Q are self mappings on a complete Menger space (X, F, t) satisfying:
(1) $A(X)\ C\ ST(X) \cap L(X) \cap M(X), A(X)\ C\ PQ(X)$.
(2) $PQ = QP, ST = TS, AQ = QA, AT = TA, LT = TL, MT = TM$.
(3) $(A, PQ), (L, ST), (A, ST), (L.M)$ are weak compatible.
(4) There exists $k \epsilon\ (0, 1)$ such that

$F_{Ax,Ay}(Kt) \geq \min\{F_{PQx,Ly}(t)\ F_{STy,Ly}(\alpha t), F_{Ay,PQx}((2-\alpha)t), \left(\frac{F_{PQx,Ly}(t) \cdot F_{Ax,PQx}(t)}{F_{PQx,STy}(t)}\right), \left(\frac{F_{STy,Ax}(t) \cdot F_{Ax,My}(t)}{F_{Ax,Ly}(t)}\right)\}$

For all x, y ϵ X , α ϵ (0, 2)and t > 0.
(5). If one of A(X), ST(X), PQ(X) is a complete sub space of X, then :
(a) A and PQ have a coincidence point.
(b) A and ST , L , M have a coincidence point.
Then self –maps A,S,T,L,M,P and Q have a unique common fixed point in X.

References

[1] Chang S. S., Cho. Y. J., and Kang S. M.. Nonlinear Operator Theory in Probabilistic Metric spaces. Nova Science Publishers, Huntington, USA, 2001.
[2] Jungck G.. Compatible mappings and common fixed points. Int. J. math. Math. Sci., 9:771:773, 1986
[3] Jungck G. and Rhoades B.E.. Fixed points for set valued functions without continuity. Indian J. Pure Appl. Math., 29:227:238,1998
[4] Menger K.. Statistical metric. Proc. Nat. Acad. (USA) 28:535:537, 1942
[5] Schweizer B. and Sklar A.. Statistical metric spaces. Pacific J. Math., 10:313:334, 1960
[6] Schweizer B. and Sklar A. Probabilistic metric spaces. Elsevier, North-Holland, New York, 1983
[7] Sehgal V. M. and Bharucha A. T.-Reid. Fixed point of contraction mappings on probabilistic metric spaces. Math. Syst. Theory, 6:97:102, 1972.
[8] Singh B. and Jain S.. A fixed point theorem in Menger space through weak compatibility. J. Math. Anal. Appl., 301:439:448,2005
[9] Singh B. and Jain S.. Fixed point theorems in Menger space through semi compatibility,. Varahmihir Journal I 117-128,2006