Common Fixed Point Theorem for Eight Mappings in Menger Space Using Rational Inequality Without Continuity

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Abstract: In this paper we establish Common fixed point theorem for eight mappings in Menger space using the notion of compatibility including rational term without continuity

Keywords: Menger space, Weak compatibility, Common fixed point.

1. Introduction

There have been number of generalizations of metric spaces. One such generalization is Menger space introduced in 1942 by Menger [4] who used distribution functions instead of nonnegative real numbers as values of the metric. This space was expended rapidly with the pioneering of Schweizer and Sklar [5,6]. This concept plays a vital role in probabilistic functional analysis, nonlinear analysis and applications [1]. In 1972, V.M. Sehgal and A.T. Bharucha-Reid [7] obtained a generalization of Banach contraction principle on a complete Menger space which is milestone in development of fixed point theory in Menger space. In 1986, Jungck [2] introduced the notion of compatible mappings in metric spaces, and this condition has further been weakened by introducing the notion of weakly compatible mappings by Jungck and Rhoades [3].

The concept of weakly compatible mappings is most general as each pair of compatible mappings is most compatible but the reverse is not true. Recently Singh and Jain [8] established a common fixed point theorem in Menger space using the concept of weak compatibility and compatibility of pair of self maps. In this paper we generalize and extend the result of Singh and Jain [9] for eight mapping opposed to six mappings in complete Menger space using the concept of compatibility including rational term without continuity.

2. Preliminaries

Definition 2.1: A probabilistic metric space (PM-space) is an ordered pair \((X, F)\) consisting of a non-empty set \(X\) and a function \(F: X \times X \to L\) where \(L\) is the collection of all distribution functions and the value of \(F\) at \((u,v) \in X \times X\) is represented by \(F_{u,v}\). The function \(F_{u,v}\) is assumed to satisfy the following conditions:

\[
\text{PM- 1) } F_{u,v}(x) = 1, \text{ for all } x > 0 \text{ if and only if } u = v
\]

\[
\text{PM- 2) } F_{u,v}(0) = 0;
\]

\[
\text{PM- 3) } F_{u,v} = F_{v,u};
\]

\[
\text{PM- 4) } F_{u,v}(x) = 1 \text{ and } F_{v,w}(y) = 1 \text{ then } F_{u,w}(x+y) = 1 \text{ for all } u,v,w \in X \text{ and } x,y > 0.
\]

Definition 2.2: A mapping \(t: [0,1] \times [0,1] \to [0,1]\) is called a \(t\)-norm if

\[
\begin{align*}
(a) & \quad t(a, 1) = a, \quad t(0, 0) = 0 \\
(b) & \quad t(a, b) = t(b, a) \quad \text{(symmetric property)} \\
(c) & \quad t(c, d) = t(a, b) \quad \text{for } c \geq a, \quad d \geq b \\
(d) & \quad t(a, b, c) = t(a, t(b, c))
\end{align*}
\]

Definition 2.3: A Menger space is a triplet \((X, F, t)\) where \((X, F)\) is a PM-space and \(t\) is a \(t\)-norm such that the inequality

\[
F_{u,w}(x+y) \geq t\{F_{u,v}(x), F_{v,w}(y)\} \quad \text{for all } u,v,w \in X \text{ and } x,y > 0
\]

Definition 2.4: Self-maps \(A\) and \(S\) of a Menger space \((X, F, t)\) are said to be weak compatible if they commute at their coincidence points i.e. if \(Ap = Sp\) for some \(p \in X\) then \(A\) and \(S\) are said to be weak compatible if they commute at their coincidence points i.e. if \(Ap = Sp\) for some \(p \in X\) then \(A\) and \(S\) are said to be weak compatible.

Lemma (1): Let \(\{x_n\}\) be a sequence in a Menger space \((X, F, t)\) with continuous \(t\)-norm and \(t(x, x) \geq x\). Suppose for all \(x \in [0,1]\) there exists \(k\) such that for all \(x > 0\) and \(n \in N\)

\[
F_{x,x_n}(x) \geq k \quad \text{for all } x > 0 \text{ and } n \in N
\]

Then \(\{x_n\}\) is a Cauchy sequence in \(X\).

Lemma (2): Let \((X, F, t)\) be a Menger space. If there exists \(k \in (0, 1)\) such that for all \(p, q \in X\)

\[
F_{p,q}(kx) \geq kF_{p,q}(x).
\]

Then \(p = q\).

In 2006, Bijendra Singh and Shishir Jain [9] introduced a generalization of Banach contraction principle in Menger space through semi-compatibility and gave the following fixed point theorem for six mappings:

Theorem: Let \(A\), \(B\), \(S\), \(T\), \(L\) and \(M\) be self-maps on a complete Menger space \((X, F, t)\) satisfying:

\[
\begin{align*}
(a) & \quad L(X) \subseteq ST(X), M(X) \subseteq AB(X) \\
(b) & \quad AB = BA, ST = TS, LB = BL, MT = TM \\
(c) & \quad Either AB or L is continuous. \\
(d) & \quad All pairs (A, B), (S, T) and (M, L) are weakly compatible.
\end{align*}
\]

Then there exists \(x \in X\) such that \(Ax = Bx = Sx = Tx = Lx = Mx\).
3. Main Results

Theorem (3.1): Let A, B, S, T, M and Q be self mappings on a complete Menger space (X, F, t) satisfying:

(3.1.1) $A(X) \subseteq ST(X) \cap L(X) \cap M(X), B(X) \subseteq PQ(X).

(3.1.2) $PQ = PQ, ST = TS, AQ = QA, BT = TB, LT = TL, MT = TM.$

Let $x_0 \in X$ be condition 3.1.1 there exists $x_1, x_2 \in X$ such that

$A x_0 = ST x_0 = L X_0 = M x_0 = x_1$ and $B x_1 = PQ x_2 = y_1.$ Inductively we can construct sequence $\{x_n\}$ and $\{y_n\}$ in X. such that $y_{2n} = A x_0 = ST x_0 = L X_{2n-1} = M X_{2n+1}$ and $y_{2n} = B x_{2n} = PQ X_{2n+2}$ for $n = 0, 1, 2, \ldots$

putting $x = x_{2n}$ and $y = x_{2n+1}$ for $x > 0$ in 3.1.4 then we have

$A x_{2n} B x_{2n+1} (K) \geq \min \{F_{PQ} x_{2n}, L x_{2n+1} (t), F_{ST} x_{2n+1} L x_{2n+1} (t), F_{BT} x_{2n+1} P Q x_{2n+2} (((2 - \alpha) t), F_{AQ} x_{2n} M x_{2n+1} (t) \right) F_{AX} x_{2n} A x_{2n+1} (t) \}

F_{y_{2n}, y_{2n+1}} (K) \geq \min \{F_{y_{2n-1}, y_{2n}} (t), F_{y_{2n}, y_{2n+1}} (at), F_{y_{2n+1}, y_{2n+1}}((-2) a t) \}

Hence $F_{y_{2n}, y_{2n+1}} (K) \geq \min \{F_{y_{2n-1}, y_{2n}} (t), F_{y_{2n}, y_{2n+1}} (at), F_{y_{2n+1}, y_{2n+1}}((-2) a t) \}$

Let $\beta \in (0, 1)$ and put $\beta = 1 - \alpha$ we get

$F_{y_{2n}, y_{2n+1}} (K) \geq \min \{F_{y_{2n-1}, y_{2n}} (t), F_{y_{2n}, y_{2n+1}} ((1 - \beta) t), F_{y_{2n+1}, y_{2n+1}}((1 + \beta) t) \}

F_{y_{2n}, y_{2n+1}} (K) \geq \min \{F_{y_{2n-1}, y_{2n}} (t), F_{y_{2n}, y_{2n+1}} (t), F_{y_{2n+1}, y_{2n+1}} (t) \}

Making $\beta \rightarrow 1$, we get

$F_{y_{2n}, y_{2n+1}} (K) \geq \min \{F_{y_{2n-1}, y_{2n}} (t), F_{y_{2n}, y_{2n+1}} (t) \}

Similarly, $F_{y_{2n+1}, y_{2n+2}} (K) \geq \min \{F_{y_{2n+1}, y_{2n+2}} (t), F_{y_{2n+2}, y_{2n+2}} (t) \}$

Therefore for all $n$ even or odd we have

$F_{y_{n}, y_{n+1}} (K) \geq \min \{F_{y_{n}, y_{n+1}} (t), F_{y_{n}, y_{n+1}} (t) \}$

Consequently, it follows that for $p = 1, 2, 3, \ldots$

$F_{y_{n}, y_{n+1}} (K) \geq \min \{F_{y_{n-1}, y_{n}} (t), \frac{1}{k^n} \}

By noting that $F_{y_{n}, y_{n+1}} (t) \rightarrow 1$ as $n \rightarrow \infty$ it follows that

$F_{y_{n}, y_{n+1}} (K) \geq \min \{F_{y_{n-1}, y_{n}} (t) \}$ for all $n \in N$ and $t > 0$.

Hence by Lemma (1). ($\{ y_n \}$ is a Cauchy sequence in X. Now suppose PQ (X) is complete. Note that the sub sequence $\{ y_{2n+1} \}$ is contained in PQ (X) call it z. Let $u \in PQ^{-1} (z)$ then PQu = z. We shall use the fact that subsequence $\{ y_{2n+1} \}$ also converges to z.

$B x_{2n+1} \rightarrow z, ST x_{2n+1} \rightarrow z, L X_{2n+1} \rightarrow z, M X_{2n+1} \rightarrow z$.

$A x_2 \rightarrow z$, PQ X_{2n+2} \rightarrow z. By putting $x = u$ and $y = x_{2n+1}$ with $\alpha = 1$ in 3.1.4

$F_{A u, B x_{2n+1}} (K) \geq \min \{F_{PQ u, L x_{2n+1} (t), F_{ST} x_{2n+1} L x_{2n+1} (t), F_{BT} x_{2n+1} P Q u (t), F_{AQ} x_{2n} M x_{2n+1} (t) \}, F_{A u, x_{2n+1}} (t) \}

Taking $n \rightarrow \infty$ we get

$F_{A u, x} (K) \geq \min \{F_{x, x} (t), F_{x, z} (t), F_{z, z} (t), (F_{x, x} (t), F_{x, z} (t), F_{z, z} (t)) \}

Thus we have

$F_{A u, x} (K) \geq F_{A u, z} (t)$

Therefore by Lemma 2 we have $Au = z$. since $PQ u = z$ thus we have $Au = PQ u = z$ that is $u$ is coincident point of A and PQ this prove 3.1.5 (a).

Since $A(X) \subseteq ST(X) \cap L(X) \cap M(X)$, $Au = z$ implies that $ST(X) \cap L(X) \cap M(X)$ Then $ST u = L u = M v = z$. By putting $x = x_{2n+2}$ and $y = v$ with $\alpha = 1$ in 3.1.4

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By putting $x = z$, $y = x$ we have

$$F_{Az} = Bz = Lz = Mz = Sz = Tz = Pz = Qz = z.$$

Thus we have $F_{Az}(z) \geq F_{Az}(t)$, Therefore by Lemma (2) we have $Qz = z$. Since $PzQz = z$ therefore $Pz = z$. By combining the above results we have $Az = Bz = Lz = Mz = Sz = Tz = Pz = Qz = z$. That is $z$ is a common fixed point of $A, B, L, M, S, T, P$ and $Q$.

Uniqueness: Let $z' (z \neq z')$ be another common fixed point of $A, B, L, M, S, T, P$ and $Q$, then $Az' = Bz' = Lz' = Mz' = Sz' = Tz' = Pz' = Qz' = z'$. By putting $x = z$ and $y = z'$ with $\alpha = 1$ in 3.1.4

$$F_{Az}(z) = F_{Az}(z'), F_{Bz}(z) = F_{Bz}(z'), F_{Lz}(z) = F_{Lz}(z'), F_{Mz}(z) = F_{Mz}(z'), F_{Sz}(z) = F_{Sz}(z'), F_{Tz}(z) = F_{Tz}(z'), F_{Pz}(z) = F_{Pz}(z'), F_{Qz}(z) = F_{Qz}(z').$$

Thus we have $F_{Az}(z) = z$, Therefore by Lemma (2) we have $z$ is a common fixed point of $A, B, L, M, S, T, P$ and $Q$. Therefore $F_{Az}(z) = z$.

Taking $n \to \infty$ we get

$$F_{Az}(z) \geq \min\{ F_{Az}(t), F_{Az}(z), F_{Az}(z'), \frac{F_{Az}(z)}{F_{Az}(z')}, \frac{F_{Az}(z')}{F_{Az}(z)} \}.$$
F_{Ax,Bz}(Kt) ≥ min\{ F_{PQz,Lz}(t), F_{STz,Lz}(t), F_{Bz,PQz}(t), \left( \frac{F_{PQz,Lz}(t) \cdot F_{Az,PQz}(t)}{F_{PQz,Lz}(t)} \right), \left( \frac{F_{STz,Lz}(t) \cdot F_{Az,Mz}(t)}{F_{STz,Lz}(t)} \right) \}.

F_{z,x}(Kt) ≥ min\{ F_{x,z}(t), F_{z,y}(t), F_{z,x}(t), \left( \frac{F_{z,y}(t) \cdot F_{z,x}(t)}{F_{z,y}(t)} \right), \left( \frac{F_{z,x}(t) \cdot F_{z,x}(t)}{F_{z,x}(t)} \right) \}.

F_{z,x}(Kt) ≥ F_{z,x}(t) \text{ by Lemma (2) } z = z'. \text{ Therefore } z \text{ is a common fixed point of } A, B, L, M, S, T, P \text{ and } Q. \text{ This completes the proof.}

Corollary 3.1.6: Let A, S, T, L, M, P and Q are self-mappings on a complete Menger space (X, F, t) satisfying:
(1) \( A(X) \subseteq ST(X) \cap L(X) \cap M(X), A(X) \subseteq PQ(X). \)
(2) \( PQ = QP, ST = TS, AQ = QA, AT = TA, LT = TL, MT = TM. \)
(3) \( (A, PQ), (L, ST), (A, ST), (L, M) \) are weak compatible.
(4) There exists \( k \in (0, 1) \) such that
\[
F_{Ax,Ay}(Kt) ≥ \min\{ F_{PQx,ly}(t), F_{Ax,PQx}(t), \left( \frac{F_{PQx,ly}(t) \cdot F_{Ax,PQx}(t)}{F_{PQx,ly}(t)} \right), \left( \frac{F_{Ax,ly}(t) \cdot F_{Ax,My}(t)}{F_{Ax,ly}(t)} \right) \}
\]
For all \( x, y \in X, \alpha \in (0, 2) \)and \( t > 0. \)
(5). If one of \( A(X), ST(X), PQ(X) \) is a complete sub space of \( X, \) then:
(a) \( A \) and \( PQ \) have a coincidence point.
(b) \( A \) and \( ST, L, M \) have a coincidence point.
Then self-maps \( A, S, T, L, M, P \) and \( Q \) have a unique common fixed point in \( X. \)

References