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On the Evaluation of the Fifth Degree Elliptic Singular Moduli

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Abstract: ***We find in a closed-algebraic radicals form the value of the fifth degree singular moduli at r in any order, knowing only two consecutive values i.e. the values at r and r/25.***

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1. Introduction

Modular equations of k_{r_0} (the elliptic singular moduli), have considered and have been discussed in the last 200 years by many great Mathematicians. They play very important role in several problems. The construction of π approximation formulas, the evaluation of the famous Rogers-Ramanujan and similar continued fractions, the solution of the quintic and sextic equation, the evaluation of the elliptic integrals in modular bases other than the classical (i.e. the cubic, the quartic and the fifth), the evaluation of the derivatives of Jacobi theta functions and many other problems of mathematics (see [1],[2],[4],[5],[6],[11],[15], [16]).

The only known solvable modular equations of k_{r_0}, where that of 2nd and 3rd degrees. The partial solution of 5-th degree modular equation presented here is a new and important result.

As application of this result we give an evaluation, in a closed form, of a quintic iteration formula for $1/\pi$, constructed by the Borwein's brothers and Bailey (see [11],[5] pg.175 and [1] pg.269). We begin with the definition of the complete elliptic integral of the first kind which is (see [3],[4],[5]):

$$K(x)=\frac{\pi}{2}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;x^2\right)=\int_0^{\pi/2}\frac{dt}{\sqrt{1-x^2\sin(t)^2}} \quad :(1)$$

It is known that the inverse elliptic nome (singular modulus or elliptic singular moduli), $k=k_r$, $k_r'=\sqrt{1-k_r^2}$, is the real solution of the equation:

$$\frac{K(k_r')}{K(k_r)}=\sqrt{r} \quad :(2)$$

with $0<k_r<1$.

In what follows we assume that $r\in\mathbb{R}_+^*$. If r is a positive rational then k_r is algebraic number. The function k_r can evaluated in certain cases exactly (see [2],[5],[9],[17]). Continuing, we define for $|q|<1$ the Ramanujan's eta function

$$f(-q)=\prod_{n=1}^{\infty}(1-q^n) \quad :(3)$$

For $|q|<1$, the Rogers-Ramanujan continued fraction (RRCF) is defined as

$$R(q)=\frac{q^{1/5}}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots \quad :(4)$$

and the following relation of Ramanujan holds (see [1],[2],[8]):

$$\frac{1}{R^5(q)}-11-R(q)^5=\frac{f(-q)^6}{qf(-q^5)^6} \quad :(5)$$

We can write the eta function f using elliptic functions. It holds with $q=e^{-\pi\sqrt{r}}$, $r>0$:

$$f(-q)^8=\frac{2^{8/3}}{\pi^4}q^{-1/3}(k_r)^{2/3}(k_r')^{8/3}K(k_r)^4 \quad :(6)$$

Also holds (see [3] pg. 488)

$$f(-q^2)^6=\frac{2k_rk_r'K(k_r)^3}{\pi^3q^{1/2}} \quad :(7)$$

Theorem 1.1 (see [3] pg. 488)

$$R(q^2)^{-5}-11-R(q^2)^5=$$
$$=\left(\frac{k_rk_r'}{ww'}\right)^2\left(\frac{w}{k_r}+\frac{w'}{k_r'}-\frac{ww'}{k_rk_r'}\right)^3 \quad :(8)$$

with $w^2=k_rk_{25r}$, $(w')^2=k_r'k_{25r}'$.

$$k_r^6+k_r^3(-16+10k_r^2)w+15k_r^4w^2-20k_r^3w^3+$$
$$+15k_r^2w^4+k_r(10-16k_r^2)w^5+w^6=0 \quad :(9)$$

Once we know k_r, its relation with w and hence with k_{25r} is given from equation (9). Hence the problem of finding

k_{25r} reduces to solve the 6-th degree polynomial equation (9), which under the change of variables $w=\sqrt{k_r k_{25r}}$, $u^8=k_r^2$, $v^8=k_{25r}^2$ reduces to the "depressed equation" (see [4] chapter 10):

$$u^6-v^6+5u^2v^2(u^2-v^2)+4uv(1-u^4v^4)=0 \quad :(10)$$

The depressed equation is also related with the problem of the solution of the general quintic equation

$$ax^5+bx^4+cx^3+dx^2+ex+f=0 \quad :(11)$$

which can reduced with a Tchirnhausen transform into Bring's form

$$x^5+ax+b=0 \quad :(12)$$

The solution of the depressed equation is a relation of the form

$$k_{25r}=\Phi(k_r) \quad :(13)$$

But such construction of the root of the depressed equation can not found in radicals (see [11]). Speaking clearly the equations (9) and (10) are not solvable in radicals. Hence we seek a way to reduce them. A way can found using the extra value of $k_{r/25}$.

In this paper we give a solution of the form

$$k_{25r}=\Phi(k_r,k_{r/25}) \quad :(14)$$

which can written more general

$$k_{25^n r_0}=\Phi_n(k_{r_0},k_{r_0/25}),\ n\in\mathbb{N} \quad :(15)$$

with $\Phi_n(x)$, known algebraic functions which we evaluate them.

2. State of the Main Theorem

Our main result is

Main Theorem

For $n=1,2,3,\ldots$ we have

$$k_{25^n r_0}=$$

$$=\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-4(k_{r_0}k'_{r_0})^2\prod_{j=1}^{n}P^{(j)}\left[\sqrt[12]{\frac{k_{r_0}k'_{r_0}}{k_{r_0/25}k'_{r_0/25}}}\right]^{24}}} \quad :(16)$$

Where the function P is in radicals known function and is given from

$$P(x)=P[x]=U\left[Q\left[U^*[x]^6\right]^{1/6}\right] \quad :(17)$$

$$P^{(n)}(x)=\left(\underbrace{P\circ\ldots\circ P}_{n-times}\right)(x),\ P^{(1)}(x)=P(x) \quad :(18)$$

The function Q is that of (30) and U, U^* are given from (33) and (34) below. The symbol "$\circ$" means iteration.

3. The Reduction of the Evaluation of the 5-th Degree Modular Equation

We give below some Lemmas that will help in the construction of proof and evaluation of function P of Main Theorem.

Lemma 3.1 (see also [6])

Let $q=e^{-\pi\sqrt{r}}$ and r real positive. We define

$$A_r:=\left(\frac{k'_r}{k'_{25r}}\right)^2\sqrt{\frac{k_r}{k_{25r}}}M_5(r)^{-3} \quad :(19)$$

then

$$R(q)=\left(-\frac{11}{2}-\frac{A_r}{2}+\frac{1}{2}\sqrt{125+22A_r+A_r^2}\right)^{1/5} \quad :(20)$$

where $M_5(r)$ is root of:

$$(5X-1)^5(1-X)=256(k_rk'_r)^2X.$$

Proof.

Suppose that $N=n^2\mu$, where n is positive integer and μ is positive real then holds that

$$K[n^2\mu]=M_n(\mu)K[\mu] \quad :(21)$$

where $K[\mu]:=K(k_\mu)$.

The following equation for $M_5(r)$ is known (see [17] pg.75) :

$$(5M_5(r)-1)^5(1-M_5(r))=256(k_rk'_r)^2M_5(r) \quad :(22)$$

Thus if we use (5),(6),(20), we get:

$$R(q)^{-5}-11-R(q)^5=\frac{f(-q)^6}{qf(-q)^6}=A_r=$$

$$=\left(\frac{k'_r}{k'_{25r}}\right)^2\sqrt{\frac{k_r}{k_{25r}}}M_5(r)^{-3} \quad :(23)$$

Solving with respect to $R(q)$, we get the result.

Let now $q=e^{-\pi\sqrt{r}}$, $r>0$ and $v_r=R(q)$, it have been proved by Ramanujan that (see [1],[2],[8],[10]):

$$v_{r/25}^5=v_r\frac{1-2v_r+4v_r^2-3v_r^3+v_r^4}{1+3v_r+4v_r^2+2v_r^3+v_r^4} \quad :(24)$$

Also from (23) it is

$$A_r=R(q)^{-5}-11-R(q)^5=\frac{f(-q)^6}{qf(-q^5)^6} \quad :(25)$$

Then from Lemma 3.1

$$v_r = R(q) = S(A_r) =$$

$$= \left(-\frac{11}{2} - \frac{A_r}{2} + \frac{1}{2}\sqrt{125 + 22A_r + A_r^2}\right)^{1/5} \quad : (26)$$

Note that a function S is defined from the 3rd equality of (26). From the above we get the following modular equation for A_r

$$A_{r/25} = v_{r/25}^{-5} - 11 - v_{r/25}^{5} = F(v_r) =$$

$$= \left(v_r \frac{1 - 2v_r + 4v_r^2 - 3v_r^3 + v_r^4}{1 + 3v_r + 4v_r^2 + 20v_r^3 + v_r^4}\right)^{-1} - 11 -$$

$$- \left(v_r \frac{1 - 2v_r + 4v_r^2 - 3v_r^3 + v_r^4}{1 + 3v_r + 4v_r^2 + 20v_r^3 + v_r^4}\right)^{-1} \quad : (27)$$

and from (26) replacing v_r in terms of A_r

$$A_{r/25} = F(v_r) = F(S(A_r)) =$$

$$= (F \circ S)(A_r) = Q(A_r) \quad :(28)$$

which after simplification with program *Mathematica* becomes

Lemma 3.2

If $q = e^{-\pi\sqrt{r}}$, $r > 0$ and

$$A_r = \frac{f(-q)^6}{qf(-q)^6}, \text{ then } A_{r/25} = Q(A_r) \quad : (29)$$

where

$$Q(x) =$$

$$\frac{(-1 - e^{\frac{1}{5}y} + e^{\frac{2}{5}y})^5}{e^{\frac{1}{5}y} - e^{\frac{2}{5}y} + 2e^{\frac{3}{5}y} - 3e^{\frac{4}{5}y} + 5e^{y} + 3e^{\frac{6}{5}y} + 2e^{\frac{7}{5}y} + e^{\frac{8}{5}y} + e^{\frac{9}{5}y}} \quad : (30)$$

and

$$y = \operatorname{arcsinh}\left(\frac{11 + x}{2}\right) \quad : (31)$$

Note that inserting (31) to (30) and simplifying, we get an algebraic function, but for simplicity and more concentrated results we leave it as it is. Consider now the following equation which appear in Lemma 3.3 below

$$\frac{X^2}{Y\sqrt{5}} - \frac{Y\sqrt{5}}{X^2} = \frac{1}{\sqrt{5}}\left(Y^3 - Y^{-3}\right) \quad : (32)$$

This equation is solvable in radicals with respect to Y and X also. One can find the solution

$$Y = U(X) = \sqrt{-\frac{5}{3X^2} + \frac{25}{3X^2 h(X)} + \frac{X^4}{h(X)} + \frac{h(X)}{3X^2}} \quad : (33)$$

where

$$h(x) = \left(-125 - 9x^6 + 3\sqrt{3}\sqrt{-125x^6 - 22x^{12} - x^{18}}\right)^{1/3}$$

The solution of (32) with respect to X is

$$X = U^*(Y) = \sqrt{-\frac{1}{2Y^2} + \frac{Y^4}{2} + \frac{\sqrt{1 + 8Y^6 + Y^{12}}}{2Y^2}} \quad : (34)$$

Lemma 3.3 (see [8])

If G_r denotes the Weber class invariant

$$A' = \frac{f(-q^2)}{q^{1/3} f(-q^{10})} = (A_{4r})^{1/6} \text{ and } V' = \frac{G_{25r}}{G_r},$$

then

$$\frac{A'^2}{V'\sqrt{5}} - \frac{V'\sqrt{5}}{A'^2} = \frac{1}{\sqrt{5}}\left(V'^3 - V'^{-3}\right) \quad : (35)$$

Note. For the Weber class invariant one can see [14],[2].

We state now our first theorem

Theorem 3.1

For the Weber class invariant holds

$$\frac{G_r}{G_{r/25}} = U\left[Q\left[U^*\left[\frac{G_{25r}}{G_r}\right]^6\right]^{1/6}\right] \quad : (36)$$

Proof.

Set

$$A = (A_{4r/25})^{1/6} \text{ and } V' = \frac{G_r}{G_{r/25}},$$

then from Lemmas 3.2 and 3.3 and from relations (29),(30),(32),(33) and (34), we have

$$\frac{G_r}{G_{r/25}} = U\left[(A_{4r/25})^{1/6}\right] = U\left[Q(A_{4r})^{1/6}\right] =$$

$$= U\left[Q\left[U^*\left(\frac{G_{25r}}{G_r}\right)^6\right]^{1/6}\right],$$

which completes the proof.

Continuing we have

$$G_r = 2^{-1/12}(k_r k_r')^{-1/12} \quad : (37)$$

hence

$$\left(\frac{k_r k_r'}{k_{r/25} k_{r/25}'}\right)^{-1/12} = U\left[Q\left(U^*\left(\left(\frac{k_{25r} k_{25r}'}{k_r k_r'}\right)^{-1/12}\right)^6\right)^{1/6}\right] : (38)$$

From the identity

$$k_{1/r} = k_r'$$

we get

$$\left(\frac{k_{1/r}k'_{1/r}}{k_{25/r}k'_{25/r}}\right)^{-1}=U\left[Q\left(U^*\left(\left(\left(\frac{k_{1/(25r)}k'_{1/(25r)}}{k_{1/r}k'_{1/r}}\right)^{-1/12}\right)^{6}\right)^{1/6}\right]^{12}: \quad (39)$$

and setting $r\to 1/r$, we lead to

Theorem 3.2
If $r\in\mathbb{Q}^*_+$, then

$$\sqrt[12]{\frac{k_{25r}k'_{25r}}{k_rk'_r}}=U\left[Q\left[U^*\left[\sqrt[12]{\frac{k_rk'_r}{k_{r/25}k'_{r/25}}}\right]^{6}\right]^{1/6}\right] : (40)$$

Hence knowing k_{r_0} and $k_{r_0/25}$, we can evaluate in closed radical form the k_{25r_0}. If we repeat the process we can find any higher or lower order of $k_{25^n r_0}$ in closed radicals form, when $n\in\mathbb{N}$ and easily get (16).

Observe that a similar formula to (16) for the evaluation of $k_{r_0/25^n}$, $n=1,2,3,\ldots$, can extracted from (39).

Example 1.
For $r=5$, we have

$$k_{1/5}=\sqrt{\frac{9+4\sqrt{5}+2\sqrt{38+17\sqrt{5}}}{18+8\sqrt{5}}}$$

$$k_5=\sqrt{\frac{9+4\sqrt{5}-2\sqrt{38+17\sqrt{5}}}{18+8\sqrt{5}}}$$

$$k_{125}=\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-(9-4\sqrt{5})P[1]^2}} \quad : (41)$$

where $P(x)=P[x]$ is that of (17) and $P[1]$ is an algebraic radical number which has too complicated form to present it here. Its minimal polynomial is

$$1-5x^2-10x^3-15x^4-22x^5-15x^6-$$
$$-10x^7-5x^8+x^{10}=0$$

The same holds also for the next

Example 2.
For $r_0=25$ it is $k_{r_0/25}=k_1=\frac{1}{\sqrt{2}}$ and

$$k_{r_0}=k_{25}=$$
$$=\frac{1}{\sqrt{2\left(51841+23184\sqrt{5}+12\sqrt{37325880+16692641\sqrt{5}}\right)}}$$

Hence

$$k_{625}=\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-(51824-23184\sqrt{5})P\left[\frac{\sqrt{5}-1}{2}\right]^{24}}} \quad : (42)$$

By this way from 2nd and 3rd degree modular equations (see [1] chapter 19), we can evaluate every k_r which r is of the form $r=4^k\cdot 9^l\cdot 25^n r_0$, when k_{r_0} and $k_{r_0/25}$ are known and $k,l,n\in\mathbb{N}$.

4. The fifth degree singular moduli and approximations to $1/\pi$.

In [5] pg.175 J.M. Borwein and P.B. Borwein consider the following sharp convergent approximation algorithm to π (see also [11])

Consider $a_0:=a_0(r_0)$, where $a_0(r)$ is the elliptic alpha function and $u_0=\sqrt[8]{1-k_{r_0}^2}$ set also $u_n=\sqrt[4]{k_{25^{n-1}r_0}}$, which now are given from (16) and are in closed form radicals. Using the Main Theorem we are able to construct approximations not depending on numerical estimates of singular moduli k_r, but from finitely exact radicals values.

Let

$$x_n:=2u_nu_{n+1}^5,\ y_n:=2u_n^5u_{n+1}$$

$$a_n:=u_n^2+5u_{n+1}^2+2x_n,\ b_n:=5u_n^2+u_{n+1}^2-2y_n,\ \gamma_n:=\frac{a_n}{b_n}$$

Finally

$$\delta_n:=4^{-1}a_n^{-1}(1-u_{n+1}^8)[5(u_n^2+x_n)+5\gamma_n(y_n-u_{n+1}^2)]+$$
$$+4^{-1}b_n^{-1}(1-u_n^8)[u_n^2+x_n+5\gamma_n(y_n-u_n^2)]$$

Then

$$\alpha_{n+1}:=5\gamma_n\alpha_n+5^{n+1}\sqrt{r_0}(\delta_n+u_{n+1}^8-\gamma_nu_n^8) \quad : (43)$$

and

$$0<\alpha_n-\pi^{-1}<16\cdot 5^n\sqrt{r_0}e^{-5^n\pi\sqrt{r_0}} \quad : (44)$$

for $r_05^{2n}\geq 1$.

Hence for every initial condition given to $r_0>0$, with k_{r_0} and $k_{r_0/25}$ known, we lead to a closed form iteration formula approximating $1/\pi$. Actually the thirteen iterations of the above algorithm give the first one billion digits of π. Note also that algorithms presented in [11],[13] are for a specific initial value and do not cover infinite values of r_0. Here if we know only k_{r_0} and $k_{r_0/25}$ we can find all u_n and construct the iteration for $1/\pi$.

Our ideas can generalized also for the 10th degree modular equation of k_r. The 10th degree modular equation of Rogers Ramanujan continued fraction is solvable in radicals and also can put in the form $v_{r/100}=\varphi(v_r)$ (see [12]). By this way one can solve with initial conditions k_{r_0} and $k_{r_0/100}$, the general 10th degree elliptic modular equation and evaluate k_{100r}, if finds an analogue of Lemma 3.3. But it is not so economical, since the modular equation of 10th degree can be splitten to that of 2nd and 5th degree.

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